Evolution and Strategic Stability: From Maynard Smith to Kohlberg and Mertens*

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A simple and intuitive condition on a set of strategy profiles guarantees that the set has a strategically stable subset. The condition arises naturally in evolutionary contexts. Journal of Economic Literature Classification Number: C72. © 1992 Academic Press, Inc.

I. INTRODUCTION

While strategic stability (Kohlberg and Mertens [4]) is mathematically elegant, it is not yet fully understood when or why it is a reasonable condition on a set of equilibria. Cho and Kreps [1, p. 220] conclude "...if there is an intuitive story to go along with the full strength of stability, it is beyond our powers to offer it here." Elaborate refinements of Nash equilibrium motivated by considerations of rational play have in general come to be viewed with increasing skepticism.

When a given game is played repeatedly by anonymous, randomly matched players from a large population of potential players, there is the opportunity for learning or "evolution" without the difficulty of supergame effects. Evolutionary stability (Maynard Smith [5]) is essentially a condition that there be no profitable entry opportunities given the actions of the incumbent population. A remarkable result (due to van Damme [8]) shows that evolutionarily stable strategies generate proper equilibria (Myerson [6]). Thus, a rather strong solution concept can be motivated not as an implication of rational play, but instead as a necessary condition for the robustness of a society's behavior in the face of small groups of entrants playing alternate strategies.

This paper presents such a result, but one that is considerably stronger in two ways. First, we work with a much weaker solution concept. Some

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333

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entrants which upset candidate evolutionarily stable strategies can be quite implausible, especially in economic environments, and so it is not clear that being robust to them is really a necessary condition for the long term viability of a strategy profile. Rather than requiring robustness to all possible entrants, we restrict attention to those that are a best response to the post entry environment. This generates a much smaller (and hopefully more plausible) set of entrants. Our concept is also weaker in that it allows for set valued solutions.

Second, we derive a stronger implication. In particular, if a set of strategy profiles satisfying the concept also has the right "shape," then it will contain a strategically stable subset. The definition of stability we use is quite broad, including all three variants used by Kohlberg and Mertens, and the versions proposed by Hillas [3]. In many examples, such a set consists of Nash equilibria which specify the same actions along the equilibrium path. If one believes that any entrant that is a best response to its post entry environment will eventually arise, then the outcome will be sustainable only if it is strategically stable.

Section II establishes notation and defines and discusses the evolutionary condition and the shape condition. We show that the conditions are implied by several existing solution concepts. Section III sets up the necessary framework for a consideration of strategic stability. Section IV states and proves the main theorem. Section V shows that in the special case of Kohlberg and Mertens' original definition of stability, the theorem can be considerably strengthened. In particular, the set of entrants against which a set needs to be tested includes only those which are (in a sense to be made precise) perfect.

II. EVOLUTION

We consider \( n \) person normal form games. Players \( i = 1, \ldots, n \) have finite pure strategy sets \( S_i \), with \( S = \prod_{i=1}^{n} S_i \). The payoff function is \( \pi = (\pi_1, \ldots, \pi_n) \). The game generated by \( S \) and \( \pi \) is denoted \((S, \pi)\). The space of mixed strategies is \( \Phi = \prod_{i=1}^{n} \Delta(S_i) \). The set of best responses to \( \sigma \in \Phi \) is \( B(\sigma) \subseteq \Phi \). The set of Nash equilibria of \((S, \pi)\) is \( N(S, \pi) \). \( D(\gamma, \mu) \) denotes the Euclidean distance between \( \gamma, \mu \in \Phi \). For \( \gamma \in \Phi \) and \( \Theta \subseteq \Phi \) closed, \( D(\gamma, \Theta) = \min_{\sigma \in \Theta} D(\gamma, \sigma) \).

To place \((S, \pi)\) in an evolutionary setting, we assume that to each player position \( i \), there is associated a population of potential player \( i \)'s. Sets of \( n \) players, one from each population, are repeatedly and randomly matched to play \((S, \pi)\). The population strategy for each position is simply the expectation of the strategy across the potential players for that position. An evolution-based solution concept looks for a population strategy profile, or
set of strategy profiles, which is robust against small groups of entering players playing alternate strategy profiles. For symmetric games, it is often appropriate to think of the matches as being \( n \) members from a single population, and to think of entry as occurring within that population.\(^1\)

Evolutionary stability \([5]\) requires a single strategy profile to have the property that for every entrant strategy different than the status quo, the entrant does worse against a population made up almost all of the status quo plus a little of the entrants than does the status quo. We weaken this in two ways. First, a set valued solution concept is introduced. Second, the entrants against which the set needs to be robust are restricted to those that are a best response to the post entry population. Formally, if the status quo population is playing \( \sigma \), the entrants \( \mu \), and the post entry population \( \eta \) (\( \eta \) a convex combination of \( \sigma \) and \( \mu \)), then \( \mu \) is an equilibrium entrant if \( \mu \in B(\eta) \). This leads us to:

**DEFINITION 1.** A closed set \( \Theta \subseteq \Phi \) is uniformly robust to equilibrium entrants (UREE) if there is a neighborhood \( Y \) of \( \Theta \) such that

\[
\forall \sigma \in \Theta, \forall \mu \in \Phi, \forall \eta = (1 - \lambda) \sigma + \lambda \mu, \lambda \in [0, 1],
\]

if \( \eta \in Y \) and \( \mu \in B(\eta) \), then \( \eta \in \Theta \).

Thus, a closed set \( \Theta \) of strategy profiles is uniformly robust to equilibrium entrants if there is a neighborhood of \( \Theta \) such that no equilibrium entrant can take the population from \( \Theta \) to \( Y \setminus \Theta \). Definition 1 is equivalent to the existence of an \( \varepsilon > 0 \) such that equilibrium entrants which move the population a distance no greater than \( \varepsilon \) are unable to take the population from \( \Theta \) to outside of \( \Theta \).\(^2\)

If a symmetric game is played by players drawn from a single population, then a symmetric version of robustness is appropriate. A set of symmetric profiles \( \Theta \) is uniformly robust to symmetric equilibrium entrants (URSEE) if there exists a neighborhood \( Y \) of \( \Theta \) such that (1) holds for symmetric entrants \( \mu \).

Uniform robustness to equilibrium entrants is closely related to the concept of equilibrium evolutionary stability (Swinkels \([7]\)). A closed set of Nash equilibria is equilibrium evolutionarily stable (EES) if there is \( \varepsilon > 0 \) such that no equilibrium entrant making up a fraction less than \( \varepsilon \) of the post entry population can take the population from within \( \Theta \) to outside of

\(^1\)The results of this paper are also easily extended to the Gilboa and Matsui \([2]\) framework, in which subsets of players may come from the same population.

\(^2\)It is clearly no stronger, since \( \{ y \mid D(y, \Theta) \leq \varepsilon \} \) is a neighborhood of \( \Theta \). Conversely, the function \( M(\sigma) = \sup\{ \varepsilon \mid D(\eta, \sigma) < \varepsilon \Rightarrow \eta \in Y \} \) is continuous on the closed set \( \Theta \), and so attains its minimum \( \varepsilon' \) at some point \( \sigma' \) in \( \Theta \). Since \( Y \) is a neighborhood, \( M(\sigma') > 0 \), and so \( M(\sigma')/2 \) serves the purpose.
Uniform robustness is less restrictive than EES in not requiring the elements of \( \Theta \) to be Nash equilibria, but more restrictive in that the size of entrants is measured by the distance the population is moved, rather than the fraction of the post entry population consisting of entrants.\(^3\) For sets with a single element, the two concepts of entrant size are equivalent.\(^4\) It is also easily seen that if a UREE set has a single element, then that element is a Nash equilibrium. Thus uniform robustness and EES agree for sets with a single element. Since evolutionary stability for asymmetric games implies EES ([7], Observation 2 and discussion following Definition 10), evolutionarily stable strategies are UREE. For two person games with a generic extensive form, EES implies UREE [7, Lemmas 8–10]. Analogous statements relate URSEE to the symmetric versions of evolutionary stability and equilibrium evolutionary stability. The reader is referred to [7] for additional details and motivation.

Restricting attention to connected UREE sets does not change the set of strategies that are contained in UREE sets. If a closed but not connected set \( \Theta \) satisfies (1) for some neighborhood \( Y \) of \( \Theta \), then any maximal closed connected subset \( \Theta' \) of \( \Theta \) will satisfy (1) for any neighborhood \( Y' \subseteq Y \) of \( \Theta' \) that does not intersect \( \Theta \setminus \Theta' \). UREE sets need not contain any Nash equilibria, but if \( \Theta \) is UREE and \( \Theta \) contains any Nash equilibrium, then \( \Theta \) contains all the Nash equilibria in that connected component.

The main theorem of this paper is that if a set of strategies is UREE and has the right "shape," then it will contain a strategically stable subset. A similar result relates URSEE to a symmetric version of stability. We now define the shape condition:

**Definition 2.** Let \( \Theta \subseteq \Theta \) be closed, and let \( Y \) be a neighborhood of \( \Theta \). A directional retract for \( \Theta \) and \( Y \) is a map \( A: \Theta \rightarrow Y \) such that

\[
\begin{align*}
(2.1) & \quad A \text{ is continuous}, \\
(2.2) & \quad A(v) = v \quad \forall v \in Y, \\
(2.3) & \quad A^{-1}(\Theta) = \Theta, \text{ and} \\
(2.4) & \quad \forall \mu \in \Phi \setminus Y, \ \exists \lambda \in (0, 1) \text{ and } \sigma \in \Theta \text{ such that } A(\mu) = (1 - \lambda) \sigma + \lambda \mu .
\end{align*}
\]

Note that an entrant which is a fraction \( \epsilon \) of the population can only move the population a fraction \( \epsilon \) of the distance toward any particular boundary of the strategy simplex. If \( \Theta \) contains a sequence of elements \( \sigma^t \) with the property that for some \( s_i \), \( \sigma^t(s_i) > 0 \) but \( \sigma^t(s_j) \rightarrow 0 \), then the distance entrants can move the population in the direction where \( s_i \) receives 0 weight falls to 0 under the definition of EES but not under that of UREE.

Let \( \{ \sigma \} = \Theta \) and let \( m \) be the minimum weight received by any strategy which receives positive weight in \( \sigma \). Since there are a finite number of strategies, \( m \) is well defined and strictly positive. It is easily verified that entrants making up a fraction \( \epsilon > 0 \) of the population can move the population at least \( m \epsilon \) in any direction which keeps the population within the simplex.
Conditions (2.1)–(2.2) are just the statement that \( A \) is a retract of \( \Phi \) to \( Y \). Condition (2.3) requires that no point outside of \( \Theta \) gets mapped to \( \Theta \). Condition (2.4) says that for each \( \mu \in \Phi \\setminus Y \), \( A(\mu) \) is intermediate between \( \mu \) and some point in \( \Theta \). If for \( \Theta \) a UREE (URSEE) set, there is some neighborhood \( Y \) of \( \Theta \) such that (1) holds and such that there is a directional retract for \( \Theta \) and \( Y \), then we say that \( \Theta \) admits a directional retract.

If a UREE set \( \Theta \) is convex, then such a mapping is easy to construct. Choose any \( \varepsilon > 0 \) such that (1) is satisfied for \( Y \equiv \{ \gamma \mid D(\gamma, \Theta) \leq \varepsilon \} \). For \( \mu \in \Phi \), define \( V(\mu) = \arg\min_{\gamma \in \Theta} D(\mu, \gamma) \). Because \( \Theta \) is closed and convex, and \( D \) strictly convex, \( V \) is uniquely defined and continuous. Let \( L(\mu) = \{ \nu \mid \nu \in Y, \exists \alpha \in [0, 1] \text{ such that } \nu = (1 - \alpha) V(\mu) + \alpha \mu \} \). Because \( L(\mu) \) is continuous and convex valued as a correspondence, and \( D \) is convex, \( A(\mu) \equiv \arg\min_{\nu \in L(\mu)} D(\mu, \nu) \) is upper hemicontinuous. Again using the strict convexity of \( D \), \( A \) is singleton valued and thus continuous. Conditions (2.2)–(2.4) are clear by construction.

Any UREE set with a singleton element is of course trivially convex. For the normal forms of generic two person extensive form games, EES sets are convex [7, Lemma 8].

Many other subsets of the strategy space admit a map with the required properties (star convex sets are an example). A more general characterization of such sets and for what games they arise would be of great interest.

### III. Stability

The idea of stability (Kohlberg and Mertens [4]) is to examine the robustness of a set of equilibria to perturbations in the underlying game. Formally, a class \( p \) of perturbations and a metric \( m \) is established. We shall call a subset \( \Theta \subseteq N(S, \pi) \) \((m, p)\)-stable if it is a minimal closed set such that every game in the perturbation class \( p \) that is close to \((S, \pi)\) under \( m \) has a Nash equilibrium close to \( \Theta \) (in the standard Euclidean distance).

For Kohlberg and Mertens' definition of stability, a perturbation is generated by a completely mixed strategy profile \( \gamma \in \Phi \), and a vector \( \delta \in [0, 1]^n \). The payoff to each pure strategy profile \( s \) in the perturbed game is the payoff in the original game when each player plays \( (1 - \delta_i) s_i + \delta_i \gamma_i \). The distance from the perturbed game to the original game is \( \max_{i=1,...,n} \delta_i \).

Hillas [3] directly perturbs the best response correspondence. He looks at games with strategy spaces \( \Phi \), and convex valued upper hemicontinuous best response correspondences that are close to that of the original game.

If in a symmetric game, the players are drawn from a single population with no distinction made between players in different positions, then it seems natural to consider only perturbations of \((S, \pi)\) that are themselves
symmetric games. We shall say that a closed subset $\Theta \subseteq N(S, \pi)$ is $(m, p^*)$-stable if it is a minimal closed subset such that every symmetric game in the perturbation class $p$ that is close to $(S, \pi)$ under $m$ has a Nash equilibrium close to $\Theta$ (in the standard Euclidean distance).

Let $(M, P)$ be the set of all metric-perturbation class pairs $(m, p)$ such that the best response correspondence for each element of $\Phi$ is upper hemi-continuous on $p$ with respect to $m$. This includes the perturbations and metrics used by Kohlberg and Mertens in defining stability, hyperstability, and full stability and by Hillas in his formulation of stability.

Several formulations of stability consider perturbations not only to the game itself but also to other games with the same reduced normal form. This presents no difficulty for our analysis. Robust sets are not affected by the addition or removal of redundant strategies.

IV. EVOLUTION AND STABILITY

We now can prove the following:

**Theorem 1.** Let $\Theta$ be UREE (respectively URSEE). If $\Theta$ admits a directional retract, then $\Theta$ contains Nash equilibria, and has a subset that is $(m, p)$-stable (respectively $(m, p^*)$-stable) for all $(m, p) \in (M, P)$.

**Proof.** We show that every game close to $(S, \pi)$ has a Nash equilibrium close to $\Theta$. The existence of a minimal closed subset of $\Theta$ having the desired property is then given by Zorn's Lemma. Choose an appropriate $Y \supseteq \Theta$ and directional retract $A$. Consider a family of perturbations of $(S, \pi)$ indexed by $k$ and written $(S, \pi)^k$, with the property that as $k \to \infty$, $m((S, \pi)^k, (S, \pi)) \to 0$. For each $k$, let $B^k$ be the best response correspondence for $(S, \pi)^k$. Define $R^k: \Theta \to \Phi$ by $R^k(\sigma) = B^k(A(\sigma))$. Because $A$ is continuous, $R^k$ inherits the relevant properties of $B^k$, and thus $R^k$ has a fixed point $\sigma^k$ for each $k$ (for symmetric games and symmetric perturbations, $\sigma^k$ can be chosen to be symmetric). Choose a convergent subsequence, and call its limit $\sigma$. Now, for each $k$, we have $\sigma^k \in B^k(A(\sigma^k))$ and thus, as $(m, p) \in (M, P)$, $\sigma \in B(A(\sigma))$. By (2.4), $A(\sigma) = (1 - \lambda) \gamma + \lambda \sigma$ for some $\lambda \in [0, 1]$, and some $\gamma \in \Theta$. As $A(\sigma) \in Y$, and $\sigma \in B(A(\sigma))$, $\sigma$ is an equilibrium entrant for a population starting at $\gamma$. Uniform robustness to equilibrium entrants implies that $A(\sigma) \in \Theta$ and thus by (2.3) and (2.2), $A(\sigma) = \sigma$. Thus for $k$ large, $\sigma^k \in Y$ and so $A(\sigma^k) = \sigma^k$ by (2.2). Thus, $\sigma^k \in N((S, \pi)^k)$.

Thus, for every sequence of perturbed games going to $(S, \pi)$, we have exhibited a sequence of Nash equilibria converging to a point in $\Theta$. This limit must be a Nash equilibrium by definition of $(M, P)$.
Sets which are stable under Hillas’ definition always include a proper element. This theorem is thus a generalization of Theorem 5 of [7].

Figure 1 illustrates why URSEE is not enough to imply \((m, p)\)-stability for symmetric games with asymmetric perturbations. This game is symmetric and has a unique evolutionarily stable strategy, \((T, L)\). \((T, L)\) is not stable: A perturbed game in which the perturbation vector \(\gamma\) puts high weight on \(B\) for player I and \(C\) for player II has \((T, C)\) as its only Nash equilibrium.

V. A Stronger Entry Restriction

Testing the population against all equilibrium entrants may seem quite strong. For example, one might suspect that entrants will not play weakly dominated strategies. In this section, we examine the extent to which Theorem 1 can be extended to cover a condition similar to perfection on entrants.\(^5\)

Given a set \(\Theta\), consider a status quo strategy profile \(\sigma\), an entrant \(\mu\), and a post entry population \(\eta\) which is a convex combination of \(\sigma\) and \(\mu\). We will say that \(\mu\) is an \(\alpha\)-perfect entrant taking the population from \(\sigma\) to \(\eta\) if \(\mu\) is completely mixed, and for all \(i = 1, \ldots, n\) and \(s_i \in S_i\), if \(\pi_i(\eta \setminus s_i) < \max_{t_i \in S_i} \pi_i(\eta \setminus t_i)\) then \(\mu_i(s_i) \leq \alpha\). We say that \(\mu\) is a perfect entrant taking the population from \(\sigma\) to \(\eta\) if there is \(\{(\sigma^\alpha, \mu^\alpha, \eta^\alpha)\}_{\alpha \in [0, 1]}\), \((\sigma^\alpha, \mu^\alpha, \eta^\alpha) \rightarrow (\sigma, \mu, \eta)\) such that for each \(\alpha, \mu^\alpha\) is an \(\alpha\)-perfect entrant taking the population from \(\sigma^\alpha\) to \(\eta^\alpha\). Following the analysis for equilibrium entrants, say a closed set \(\Theta \subseteq \Phi\) is uniformly robust to perfect entrants (URPE) if there is a neighborhood \(Y\) of \(\Theta\) such that for all \(\sigma \in \Theta\), if \(\mu\) is a perfect entrant taking the population from \(\sigma \in \Theta\) to \(\eta \in Y\), then \(\eta \in \Theta\).\(^6\)

\(^5\)The analysis in a preliminary version of this paper, in which the condition on entrants was similar to properness, was flawed.

\(^6\)This definition allows the status quo strategy to differ along the sequence of \(\varepsilon\)-perfect entrants. An alternative definition would hold the status quo strategy fixed. It is debatable which is more natural: should the “trembles” include uncertainty only about other entrants’ actions, or should it include uncertainty about the status quo population as well? The current formulation is chosen for ease of analysis.
URPE sets which are subsets of Nash equilibria need not exist (Fig. 5 of [7] provides an example) although, of course, a standard application of Zorn's Lemma shows the existence of minimal closed subsets of $\Phi$ which are URPE.

For general $(m, p) \in (M, P)$, an URPE set $\Theta$ need not contain an $(m, p)$-stable set even if $\Theta$ admits a directional retract. Essentially, this is because the perfect equilibrium correspondence need not be upper hemicontinuous as payoffs are changed. As an example, in the game of Fig. 2, $\{(T, L)\}$ is URPE. However, the only hyperstable set of the game is $T \times P_2$.

The theorem does go through in the special case of Kohlberg and Mertens' stability.

**Theorem 2.** Let $\Theta$ be URPE. If $\Theta$ admits a directional retract, then $\Theta$ has a subset that is Kohlberg–Mertens stable.

**Proof.** Choose an appropriate $Y \supset \Theta$ and directional retract $A$. Let $\{\gamma^k\}_{k \in \mathbb{N}}$ be a sequence of mixed strategy profiles, and let $\{\delta^k\}_{k \in \mathbb{N}}$ be a sequence of vectors from $[0, 1]^n$, with $\max_{i=1, \ldots, n} \delta^k_i \to 0$. For each $k$, let $(S, \pi^k)$ denote the game in which strategy spaces are $S$, and the payoff vector when $\sigma$ is played is defined by $\pi^k(\sigma_1, \ldots, \sigma_n) = \pi((1 - \delta^k_1) \sigma_1 + \delta^k_1, \ldots, (1 - \delta^k_n) \sigma_n + \delta^k_n)$. For $\alpha \in ((0, 1)$, and $k \in \mathbb{N}$, define $P^k_\alpha : \Phi \to Z^\Phi$ by

\[
P^k_\alpha(\sigma) = \left\{ \mu \left| \begin{array}{l}
(1) \mu_i(s_i) \geq \frac{\alpha}{2 |S_i|} \forall i = 1, \ldots, n, \forall s_i \in S_i \\
(2) s_i \notin B^k_\alpha(\sigma) \Rightarrow \mu_i(s_i) \leq \alpha \forall i = 1, \ldots, n, \forall s_i \in S_i
\end{array} \right. \right\}.
\]

Compose $P^k_\alpha$ with $A$ to define $R^k_\alpha(\cdot) = P^k_\alpha(A(\cdot))$.

Because $P^k_\alpha$ is an upper hemicontinuous, non-empty, convex valued correspondence, and $A$ is a continuous function, $R^k_\alpha$ has a fixed point $\mu^k_\alpha$ for each $\alpha$ and $k$. For any $v \in \Phi$, let $F(v) \in \Theta$ be such that $A(v)$ is a convex combination of $v$ and $F(v)$. For $v \in \Phi \setminus Y$, such an $F(v)$ exists by (2.4). For $v \in Y$, $v = A(v)$, and so any element of $\Theta$ will do (since $v$ is expressible as a convex combination of any element $\Phi$ and itself). Then, $\mu^k_\alpha$ is an $\alpha$-perfect entrant in $(S, \pi^k)$ taking the population from $F(\mu^k_\alpha) \in \Theta$ to $A(\mu^k_\alpha)$.

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**II**

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**Figure 2**
For each \( k \), choose a subsequence of \( \{ \mu^k \}_{k \in \mathbb{N}} \) such that \( \mu^k \) converges to some \( \mu^k \). Choose a convergent subsequence of \( \{ \mu^k \}_{k \in \mathbb{N}} \) with limit \( \mu \). We will show that \( \mu \) is a perfect entrant taking the population from some \( \sigma \in \Theta \) to \( A(\mu) \).

So, choose \( \alpha > 0 \). Then, there exists \( k \) such that \( D(\mu, \mu^k) < \alpha/3 \) and \( \delta_i^k < (\alpha/3n) \) for \( i = 1, \ldots, n \). Using the continuity of \( A \), \( k \) can be chosen such that in addition \( D(A(\mu), A(\mu^k)) < \alpha/3 \). Let \( \zeta^* \) be an \( \alpha/2 \)-perfect entrant taking the population from \( F(\zeta^*) \) to \( A(\zeta^*) \) in \( (S, \pi^k) \) such that \( D(\mu^k, \zeta^*) < \alpha/3 \) and \( D(A(\mu^k), A(\zeta^*)) < \alpha/3 \). Since there is a convergent subsequence of \( \{ \mu^k \}_{k \rightarrow 10} \) with limit \( \mu^k \), since \( A \) is continuous, and since an \( \alpha \)-perfect entrant is also an \( \alpha' \)-perfect entrant for any \( \alpha' > \alpha \), such an entrant exists.

For any \( v \in \Phi \), let \( \tilde{v} \) be defined by \( \tilde{v}_i = (1 - \delta_i^k) v_i + \delta_i^k \gamma_i^k \), \( i = 1, \ldots, n \). Then, by the definition of \( \pi^k \), \( \pi(\tilde{v}) = \pi^k(\tilde{v}) \). It is easily shown that \( D(\pi^k, \pi(\tilde{v})) < (\alpha/3n) \) for \( i = 1, \ldots, n \).

Now, assume that \( s_i \notin B_i(\tilde{v}(\zeta^*)) \). Then, \( s_i \notin B_i(\tilde{v}(\zeta^*)) \), and so
\[
\zeta_i^*(s_i) \leq \frac{\alpha}{2}.
\]

But,
\[
\zeta_i^*(s_i) = (1 - \delta_i^k) \zeta^*_i(s_i) + \delta_i^k \gamma_i(s_i) \leq (1 - \delta_i^k) \zeta^*_i(s_i) + \delta_i^k \leq (1 - \delta_i^k) \frac{\alpha}{2} + \delta_i^k,
\]
and so as \( \delta_i^k < (\alpha/3n) < (\alpha/2) \),
\[
\bar{\zeta}_i^*(s_i) < \alpha.
\]

Thus, \( \zeta^* \) is an \( \alpha \)-perfect entrant in \( (S, \pi) \) taking the population from \( F(\zeta^*) \) to \( A(\zeta^*) \).

Take a sequence \( \alpha \to 0 \). By (2), \( \tilde{v}(\zeta^*) \) converges to \( A(\mu) \). By construction, \( D(\mu, \zeta^*) \leq D((\mu, \mu^k) + D(\mu^k, \zeta^*) < 2\alpha/3 \), and so \( \zeta^* \) converges to \( \mu \). Take a subsequence such that \( F(\zeta^*) \) also converges. Now, by definition of \( F \), each \( F(\zeta^*) \in \Theta \) and so as \( \Theta \) is closed, \( \lim_{x \to 0} F(\zeta^*) \in \Theta \). As \( \alpha \to 0 \), \( \max_{i=1,\ldots,n} \delta_i^k \to 0 \), and so \( D(\tilde{v}, \tilde{v}) \leq \sum_{i=1,\ldots,n} \delta_i^k \to 0 \) for any \( v \in \Phi \). Thus, \( \lim_{x \to 0} F(\zeta^*) = \lim_{x \to 0} F(\zeta^*) \), \( \lim_{x \to 0} A(\zeta^*) = \lim_{x \to 0} A(\zeta^*) = \mu(\mu) = \mu \), and \( \lim_{x \to 0} \zeta^* = \lim_{x \to 0} \zeta^* = \mu \). So, \( \mu \) is a perfect entrant taking the population from \( \lim_{x \to 0} F(\zeta^*) \in \Theta \) to \( A(\mu) \in \Theta \).
$\Theta$, and so $A(\mu^k_s) = \mu^k_s$ for $\mu^k_s$ sufficiently close to $\mu$. Thus for $k$ large, $\mu^k$ is a perfect (and so Nash) equilibrium of $(S, \pi^k)$. 

Since $\mu$ is a perfect equilibrium, it is corollary to the preceding that URPE sets which admit a directional retract always contain a perfect element. Using the techniques of the preceding theorems, it is fairly easily shown that URPE sets which admit a directional retract also contain a proper element:

**Theorem 3.** Let $\Theta$ be URPE. If $\Theta$ admits a directional retract, then it contains a proper element.

**Proof.** (sketch) Choose a convergent sequence of points $\mu^x$ which are completely mixed, and such that for all $i = 1, \ldots, n$ and $s_i, t_i \in S_i$, if $\pi_i(A(\mu^x) \setminus s_i) < \pi_i(A(\mu^x) \setminus t_i)$ then $\mu^x_i(s_i) \leq \alpha \mu^x_i(t_i)$. Such points exist for $\alpha \in (0, 1)$ by a construction analogous to that in the proof of Theorem 2. Call their limit $\mu$. Evidently, $\mu$ is a perfect entrant taking the population from some $\sigma$ in $\Theta$ to $A(\mu) \in Y$. Thus, $A(\mu) \in \Theta$, and so $A(\mu^x) = \mu^x$ for $\alpha$ sufficiently small. But, then $\mu^x$ is $\alpha$-proper for $\alpha$ sufficiently small and so $\mu$ is proper.

**References**