EFFICIENCY OF LARGE PRIVATE VALUE AUCTIONS

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We consider discriminatory and uniform price auctions for multiple identical units of a good. Players have private values, possibly asymmetrically distributed and for multiple units. Our setting allows for aggregate uncertainty about demand and supply. In this setting, equilibria generally will be inefficient. Despite this, we show that such auctions become arbitrarily close to efficient if they are “large,” and use this to derive an asymptotic characterization of revenue and bidding behavior.

KEYWORDS: Auctions, efficiency, asymptotic efficiency, large auctions.

1. INTRODUCTION

AUCTION THEORY IS IMPORTANT for two very different reasons. First, auctions are a pervasive feature of the economy: Oil leases, T-bills, paintings, wine, cattle, used cars, real estate, and new share issues are all sold to varying degrees by auction. The FCC spectrum auctions have excited considerable interest (Cramton (1995), McMillan (1994)).

Second, auctions, broadly defined, are an excellent model of price formation in other market settings, an idea dating back to Walras. However, Walras’ auctioneer is rather unsatisfying: he announces prices, aggregates net demands at those prices, and then announces new prices until such time as supply and demand balance. Only then do actual trades take place. So, while prices play an allocative role, the process by which individual actions translate into prices is not explained. To come closer to such an explanation, one would like a model in which players take actions not knowing the true state of demand and supply, and prices and allocations are a function of these actions. In short, one would like a real auction.

Reflecting their importance, auctions have received a great deal of attention from economic theorists. Among their achievements are equilibrium characterizations in a variety of settings, results about revenue equivalence and revenue rankings, and insight into optimal auction design.

Most of this work takes place under strong assumptions: Bidders desire a single unit, valuations are symmetrically and nonatomically distributed, and focus is placed on symmetric equilibria. However, in many auctions, and in particular when viewing auctions as a metaphor for other price formation

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processes, players desire more than one unit. And, in most settings asymmetry would seem more the norm than the exception. Some participants in an art auction are wealthier than others. An oil company with idle drilling equipment will have a higher incentive to obtain a lease.

Relaxing these assumptions has significant implications. Single unit discriminatory auctions (auctions in which winners pay their bids) cease to be efficient when values are asymmetric. Essentially, bidders who know that their opponents are likely to have high values will bid more aggressively than bidders who believe their opponents have low values. This is so as they have less to lose in terms of paying more in situations where they would already have won with a lower bid. With multiple unit demands, even symmetric discriminatory auctions are inefficient, because differently ranked bids by a single player face different environments. Finally, consider uniform price auctions, in which $k$ objects are sold at a price between the $k$th and $k + 1$st submitted bids. If, for example, price is the $k + 1$st bid, then with single unit demands, it is dominant to bid one's valuation. With multiple unit demands, changing one's second bid may affect not only whether a second object is won, but also the price paid when only one object is won. This again leads to inefficiency. It also becomes extremely difficult to characterize equilibria in any detail, and harder still to solve for them. These problems are troubling for applications of auction theory to actual auctions. The loss of efficiency is especially troubling when considering other price formation processes.

In many auctions, and especially when considering auctions as metaphors for other price formation processes, there are many potential buyers and many units for sale. So, it is interesting to know the properties of large auctions. In this paper, we consider auctions with a large number of players with independent and possibly asymmetric private values for one or more units of a homogeneous good. Our setting is such that even with many players and objects, aggregate uncertainty about demand or supply may but is not required to persist. Within this setting, we show that equilibria are asymptotically efficient and have a familiar limit form.

The key insight driving these results is very simple: while players’ values may come from very different distributions, their environments, and thus their optimal behavior with any given valuation, may be very similar. Any two players $i$ and $j$ have all players other than $i$ and $j$ in common as competitors. Thus, if the set of players is large, the difference between $i$ and $j$’s strategic environments “ought” to be small, so that $i$ and $j$ win with similar probabilities if they bid $b$. And, if $i$ is small relative to the overall market, whether $i$ wins with $b$ “ought” not to

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2 To see these arguments formalized, see Maskin and Riley (2000), or Lebrun (1999).
4 See Marshall et al. (1994) for several numerical examples.
5 See Jackson, Simon, Swinkels, and Zame (2000), and Jackson and Swinkels (2000) for existence in a class of auctions covering all those considered here, except that values are assumed to be nonatomic.
depend too much on whether \( b \) is \( i \)'s highest or \( h \)th highest bid.

We term this property—that as the auction grows large, both the identity of the bidder and the rank of a bid among that player's bids cease to affect the probability of a bid being among the winners—*asymptotic environmental similarity* (AES). To establish conditions under which AES holds, we identify conditions under which it becomes very unlikely that the number of bids above any given \( b \) is within a small interval around the number of objects. But then, the actions of an individual opponent, or whether \( i \) has submitted none or several bids above \( b \), has little impact on whether \( b \) wins. We show that this holds as long as there is a little uncertainty about either supply or demand.\(^6\)

It is not obvious that AES implies asymptotic efficiency, since close-by decision problems can have very different maximizers. The key for the discriminatory case is to find changes in bids that \( j \) feels strongly enough about so that \( i \), facing approximately the same problem, has the same preferences. For the uniform price case, the key is to show that because of AES, a player can more or less ignore the effect of her own bidding on prices. Hence, if the market price is likely to be much lower than the value of an unfilled unit of her demand, then she has an incentive to change her set of bids. Thus, AES implies asymptotic efficiency.

Since in the limit objects are allocated efficiently, an asymptotic version of the revenue equivalence result holds: expected revenue per unit converges to the expectation of the \( k + 1 \)st value, i.e., to the marginal valuation in an efficient allocation. This is the natural analog to the revenue in a standard auction.

Finally, since we have tied down both the limit allocations and the limit surplus, we can back out what equilibrium bids must be in the limit. In a discriminatory auction the optimal bid on a unit of demand with value \( v \) converges to the expectation of the \( k + 1 \)st value conditional on the \( k + 1 \)st value being less than \( v \). Similarly, price in the uniform auction converges in probability to the market clearing price in the analog competitive market. So, over relevant ranges of values, bids in uniform auctions converge to truth telling.\(^7\) Thus, while exact equilibria of these auctions are nearly impossible to calculate, limiting behavior is both very simple and accords well with what we have learned in the simple settings typically studied.

The order in which we derive our results is unusual. A natural plan of attack in understanding large auctions would be to write down equilibria for the finite case, and then take limits to derive properties such as asymptotic efficiency.\(^8\) This is infeasible because equilibria for a setting of the generality of this paper are very poorly understood in the finite case. So, we go the other way around: Using AES, we derive a minimal characterization of equilibria that allows us to derive limit efficiency. Limit efficiency ties down limit revenue. Finally, knowing

\(^6\) See Al-Najjar and Smorodinsky (1997) for a related discussion of situations in which uncertainty leads each player to have vanishing “influence.” The connection to that paper is strongest for the case of uniform auctions.

\(^7\) We do not rule out some “irrelevant” distortions in bidding behavior.

\(^8\) This is the approach in Katzman (1999), Pesendorfer and Swinkels (1997), and Wilson (1976).
limit revenues and limit allocations allows a characterization of limit bidding behavior.

In large number situations, small deviations from optimal play might create large aggregate deviations from equilibrium predictions. So, we consider $\epsilon$-equilibria. For each $\epsilon$, there is an $\epsilon$-equilibrium of $A^n$ for each type of auction considered. For $n$ large enough, one such equilibrium is for players to follow the simple bidding behaviors discussed above. An $\epsilon$ version of our efficiency result also holds: small deviations from equilibrium play result in small deviations from efficiency.

Swinkels (1999) considers discriminatory auctions in a setting similar to this one. The setting of that paper is less general in that it does not allow limit uncertainty about aggregate demand and supply. On the other hand, arguments based on AES play no role. Rather, the law of large numbers is the driving force. Nor, except in a degenerate sense, are we able to draw the connection between limit equilibria and the standard equilibria of simple auctions. Rather, we show that in the limit, bidders are price takers at the competitive price. Note also that our Walrasian auctioneer would have little trouble in this setting since in the limit, the market clearing price is known a priori. In contrast, in this paper the market clearing price can be uncertain even if the auction is large.

Section 2 deals with preliminaries. Section 3 discusses AES. Section 4 makes the link between AES and asymptotic efficiency for discriminatory auctions, and Section 5 for uniform auctions. Section 6 provides an asymptotic characterization of revenues and surplus. This allows us to back out the limit bid functions. Section 7 discusses $\epsilon$-equilibria. Section 8 concludes. The Appendix contains proofs omitted from the main body of the paper.

2. THE AUCTION FRAMEWORK

We consider a sequence $\{A^n\}_{n=1}^{\infty}$ of auctions. The $n$th auction, $A^n$, is defined as follows.

Players: There are $n$ potential buyers, $1, \ldots, n$.

Demands: Each player can desire up to $m$ objects, where $m$ does not depend on $n$. Player $i$’s valuation in $A^n$ is $v^n_i = (v^n_{i1}, \ldots, v^n_{im})$. Player $i$ places marginal value $v^n_{ih}$ on an $h$th object, and so value $\sum_{h=1}^{H} v^n_{ih}$ on $H$ objects. $v^n_{ih}$ is drawn according to a probability measure $G^n_i$ on $\mathbb{R}^m$. We assume the following:

(A1) BOUNDED VALUES: There is $\bar{n} < \infty$ such that $G^n_i([0, \bar{n}]^m) = 1$ for all $i$ and $n$. For convenience only, $\bar{n} \geq 1$.

The results also differ in that results can be obtained even with a fixed number of objects and a growing number of bidders. Here, we need that (in expectation) a positive fraction of the market is supplied.

A similar point applies to the double auction setting considered by e.g., Rustichini et al. (1994).

Much of this set-up is common with Swinkels (1999).

Our results extend to the case that $m/n \to 0$ sufficiently quickly in $n$. 

(A2) **Diminishing Marginal Utility:** Let \( V = \{ v \in [0, \bar{v}]^m | v_h \geq v_{h+1}, h = 1, \ldots, m-1 \} \). Then, \( G(V) = 1 \).

(A3) **Independent Values Across Players:** \( G^n_i, i = 1, 2, \ldots, n \) are independent. Of course, the various values of any given player will be correlated.

(A4) **No Asymptotic Atoms:** For each \( h \), let \( V_h \) be the subset of \( V \) in which the first \( h \) elements are nonzero, while the remaining elements are zero. (So in particular, \( V_0 = \{0\} \), where 0 is the \( m \) vector with all element 0.) Let \( \mathcal{M}_h \) be the probability measure which is uniform on \( V_h \), and let \( \mathcal{M} = (1/(m+1)) \sum_{h=0}^{m} \mathcal{M}_h \). Then, there exists \( z < \infty \) such that for each Borel set \( W \subseteq V \),

\[
\limsup_{n \to \infty} \frac{G^n_i(W)}{n} < z \mathcal{M}(W).
\]

By taking \( \mathcal{M} = (1/(m+1)) \sum_{h=0}^{m} \mathcal{M}_h \), we allow it to be a positive probability event that a player values \( H < m \) objects, but impose that the first \( H \) values do not fall with positive limit probability in a Lebesgue measure zero subset of \( \mathbb{R}^H \). Assumption (A4) rules out that in the limit, a positive fraction of values are expected to fall in some arbitrarily small region. It does not rule out atoms in the individual \( G^n_i \). For example, each \( G^n_i \) can put probability 1 on some particular value vector, as long as the atoms for various players do not pile up as \( n \) grows large.

A useful implication of (A4) is the following. For \( x \in [0, \bar{v}] \), and \( \varepsilon > 0 \), let

\[
W(x, \varepsilon) = \{ v \in V | v_h \in [x, x + \varepsilon) \text{ for some } h \}.
\]

Then, there is \( z' < \infty \) such that for all \( \varepsilon > 0 \) and \( x \in (0, \bar{v}) \),

\[
\limsup_{n \to \infty} \frac{\sum_{i=1}^{n} G^n_i(W(x, \varepsilon))}{n} \leq z' \varepsilon.
\]

While we allow players not to desire any objects, we do require that in expectation a positive fraction of players wish to own an object:

(A5) **Relevance:** There is \( Q < 1 \) such that for all \( n \),

\[
\sum_{i=1}^{n} \frac{G^n_i(0)}{nm} < Q.
\]

To allow aggregate uncertainty about demand, we introduce a probability measure \( \rho^n \) on \( \{1, \ldots, n\} \) for each \( n \). A number \( s^n \) is drawn according to \( \rho^n \). Each subset of size \( s^n \) of the players \( \{1, \ldots, n\} \) is chosen with equal probability to be active. The set of active bidders is denoted \( X^n \).\(^{13}\) We do not exclude degenerate \( \rho^n \).

\(^{13}\) This construction is used by Deneckere and Peck (1995). Its role is discussed in Section 6.2.
Supply: There are \( kn \) identical indivisible objects available for sale, determined according to a (possibly degenerate) probability measure \( \mu^n \) on \( \{1, \ldots, nm\} \). There is \( \tau > 0 \) such that for all \( n \), \( E(k^n) \geq \tau n \). \( k^n \) and \( X^n \) are independent of each other and of the various \( v_i^n \)'s.

Active players simultaneously submit bids \( b \) from \( B = \{b \in \mathbb{R}^m|b_h \geq b_{h+1}, h = 1, 2, \ldots, m-1, b_m \geq 0\} \). Inactive players are taken to submit \( m \) bids of 0. \( k^n \) is then determined according to \( \mu^n \). One object is awarded for each bid that is unambiguously among the \( k^n \) highest. If there are less than \( k^n \) bids above \( b \) but more than \( k^n \) bids of \( b \) or more, one of the remaining objects is allocated at random to a player with unfilled bid \( b \). This is repeated until the objects are gone.\(^{15}\) We rule out reserve prices.

Strategies: Strategies are measurable mappings from valuation vectors to bids. Payment Rules: In a discriminatory auction, if \( i \) is awarded \( h \) objects, then she pays the sum of her \( h \) highest bids. In a uniform price auction players pay \( p \) for each unit awarded. The price \( p \) is a continuous and weakly increasing function of the \( k^n \) and \( k^n + 1 \)st submitted bids and lies between these bids. Examples satisfying these assumptions include the lowest accepted bid and highest rejected bid auctions. We will write \( A^n_D \) when we wish to emphasize that we are referring to discriminatory auctions, and similarly \( A^n_U \) for uniform auctions.

Payoffs: The payoff to a bidder is the value of objects received less payments made. The seller puts no value on the objects. His payoff is thus the sum of payments received. All players are risk neutral.

Equilibrium: We consider Nash equilibrium in nonweakly dominated strategies.\(^{16}\) Write \( e^n \) for an equilibrium of \( A^n \).

For each \( n \), for \( v \in V, b \in B \), and for given \( (A^n, e^n) \) define \( S_i^n(v, b) \) as \( i \)'s payoff when he has value \( v \) and submits bids \( b \). In a minor abuse of notation, let

\[
S_i^n(v) = \max_{b \in B} S_i^n(v, b)
\]

be \( i \)'s expected surplus when he behaves optimally with \( v \). Let \( BR_i^n(v) = \arg\max_{b \in B} S_i^n(v, b) \) be the set of optimal bids with \( v \in V \).

Let \( y^a_i(k) \) be the \( k \)th order statistic on values for active players other than \( i \). Let

\[
K_{ih}^a(x) = \Pr(y_{ih}^a(k^n - h + 1) \leq x).
\]

At points of continuity, \( K_{ih}^a(v_{ih}^a) \) is the probability that \( i \in X^n \) efficiently receives an \( h \)th object.

For \( b \geq 0 \), let \( B^n(b) \) be the random variable giving the number of bids strictly above \( b \) given \( (A^n, e^n) \). For \( M \subseteq \{1, \ldots, n\} \), \( B_M^n(b) \) denotes the number of bids

\(^{14}\) If \( k^n \geq nm \), then all players have all of their demand filled, and so actual surplus equals feasible surplus automatically. So, all \( k^n > nm \) are equivalent to \( k^n = nm \).

\(^{15}\) If the tie breaking rule reflected how many bids of \( b \) a player made, then a player who only wanted one object might still submit several bids of \( b \) to help win in the event of a tie.

\(^{16}\) For expositional simplicity, we require payoff maximizing nonweakly dominated bids for each realization of values rather than just with probability one.
above $b$ by players in $M$ and $B^n_{-M}(b)$ by players not in $M$.

Fix $(A^n, e^n)$. An outcome specifies realizations for $v^n_1, \ldots, v^n_n, k^n$ and $X^n$, and an allocation of the $k^n$ objects among $X^n$. Define actual surplus as

$$a^n = \sum_{i \in X^n} \sum_{1 \leq h \leq H_i} v^n_{ih},$$

where $H_i$ is the number of objects $i$ wins. Define feasible surplus as

$$f^n = \sum_{i \in X^n} \sum_{1 \leq h \leq J_i} v^n_{ih},$$

where $J_i$ is the number of objects for which $i$'s value is among the $k^n$ highest in $X^n$. Payments do not enter these calculations, they are transfers.

**DEFINITION 2.1:** $(A^n, e^n)$ is asymptotically ex-ante efficient if $E(a^n)/E(f^n) \to 1$. It is asymptotically ex-post efficient if $E(a^n/f^n) \to 1$.

Swinkels (1999) gives an example showing that neither condition implies the other. Because players bid before they know the realization of the auction, our results are directly in terms of ex-ante efficiency. However, assume that asymptotically, no positive measure set of states contributes a vanishing fraction of ex-ante feasible surplus. Then if the auction performs poorly with positive probability, and so fails ex-post efficiency, then it also fails ex-ante efficiency. Formally, we have the following lemma.

**LEMMA 2.2:** Assume that for all $e > 0$, there is $\delta > 0$ and $n^* < \infty$ such that for all $n > n^*$, $Pr(k^n > \delta E(k^n)) > 1 - e$ and $Pr(s^n > \delta E(s^n)) > 1 - e$. Then, asymptotic ex-ante efficiency implies asymptotic ex-post efficiency.

3. ASYMPTOTIC ENVIRONMENTAL SIMILARITY

Fix $(A^n, e^n)$. For $h = 0, 1, \ldots, m + 1$, $i = 1, \ldots, n$, and $x \geq 0$ let

$$P^n_h(x) = Pr(B^n_i(x) \leq k^n - h)$$

be the cumulative for the $k^n - h + 1$th highest bid by $i$'s opponents. Let $\bar{B} = \{b \in B | P^n_{ih}(\cdot)$ is continuous at $b, \forall i, h\}$. For $b \in \bar{B}$, $P^n_{ih}(b)$ is the probability that $i$ wins an $h$th object with bid $b$.\(^{17}\)

**DEFINITION 3.21:** $(A^n, e^n)$ satisfies asymptotic environmental similarity (AES) if

$$(3.1) \quad \left| P^n_{ih}(x) - P^n_{ih'}(x) \right| \to 0$$

uniformly over all $i, j, h, h'$ and $x$.

AES will often be driven by exogenous noise. In that case, the $P^n_{ih}(\cdot)$ functions grow close regardless of the choice of strategies. We term this strong AES.

\(^{17}\) If $b \notin \bar{B}$ then one must adjust for the probability of winning at ties.
For discriminatory auctions, we need (3.1) to hold only for \( h, h' \in \{1, \ldots, m\} \) as opposed to \( h, h' \in \{0, \ldots, m + 1\} \). However, we have not found interesting conditions under which (3.1) holds for \( h, h' \in \{1, \ldots, m\} \) but not also for \( h, h' \in \{0, \ldots, m + 1\} \).

For uniform price auctions the cross player comparison is less important. What matters is that \(|P^n_{ik}(x) - P^n_{jk}(x)| \rightarrow 0\) uniformly over all \( i, h, h', x \). We term this *individual AES*.

For both types of auction, \( i \)'s payoff given \( v \) and \( b \) does not depend on bids by \( i \)'s opponents outside of the \( k^n - m \) through \( k^n + 1 \). So \( i \)'s expected payoff, \( S^n(v; b) \), is completely characterized on \( \bar{B} \) by the functions \( P^n_{ij}(x) \).

Given Lemma 3.2, one route to AES is to find conditions that guarantee that \( \Pr(k^n - 2m \leq B^n_{ij}(x) \leq k^n) \) grows small in \( n \). A little exogenous noise is one way to achieve this. We consider three sorts.

**Definition 3.3:** Let \( \overline{\mu} = \max_{k=0, \ldots, nm-1} \mu^n(k) \). \( \{A^n\} \) has *asymptotically diffuse supply* if \( \lim_{n \rightarrow \infty} \overline{\mu} = 0 \).

As an example, in Treasury auctions, regular bidders are not sure of the exact demand by “noncompetitive” bidders (small bidders who are allowed to request a number of units at the average price paid by regular bidders), and hence the number of units available to regular bidders. Treasury auctions do not fit our model well on some other dimensions, in particular because of the assumption of private values, and because some participants may demand a substantial fraction of the total supply.

**Definition 3.4:** \( \{A^n\} \) has *idiosyncratic demand uncertainty* if there is \( \delta > 0 \) such that for each \( n, X^n, \) and \( \iota \in X^n \), there is a further probability of at least \( \delta \) of \( i \) not participating in the auction, where this event is independent across players.

One way to guarantee this is to assume that there is a probability at least \( \delta \) that each player \( i \) has all values equal to 0 and that players with all values zero
do not participate. Another is to assume that when \(i \in X^n\), there is a further probability of at least \(\delta\) of \(i\) missing the auction, say by oversleeping. As before, we treat \(i\) as being present, but with \(v_i = b_i = 0\).

**Definition 3.5:** \(\{A^n\}\) has aggregate demand uncertainty if for all \(\epsilon > 0\), there is \(\delta > 0\) and \(n^* < \infty\) such that for all \(n > n^*\) and \(\hat{s}\),

\[
\Pr\left(s^n \in ((1 - \delta)\hat{s}, (1 + \delta)\hat{s})\right) < \epsilon.
\]

So, with aggregate demand uncertainty, the number of active bidders is unlikely to be within a small percentage of \(\hat{s}\) for any given \(\hat{s}\). \(^{18}\)

Any of these is enough for strong AES. If supply is diffuse, then for any given \(B^n_{-ij}(b), k^n\) is unlikely to be between \(B^n_{-ij}(b)\) and \(B^n_{-ij}(b) + 2m\). With either idiosyncratic or aggregate demand uncertainty, \(B^n_{-ij}(b)\) will become diffuse, and so unlikely to be between \(k^n - 2m\) and \(k^n\) for any given \(k^n\). Formally we have the following theorem.

**Theorem 3.6:** Assume \(\{A^n\}\) has at least one of asymptotically diffuse supply, idiosyncratic demand uncertainty, or aggregate demand uncertainty. Assume also that for all \(K > 0\), \(\lim_{n \to \infty} \sum_{k=1}^{K} \mu^n(k) = 0\). \(^{19}\) Then, \(\{A^n\}\) satisfies strong AES.

AES need not imply aggregate uncertainty. For example, let \(k^n\) be uniform on the integers in \((n/2 - \sqrt{n}, n/2 + \sqrt{n})\). Then, supply is asymptotically diffuse, but the ratio of supply to the number of bidders converges with certainty to 1/2. Rather, the point is that when AES holds, the auction will be asymptotically efficient whether or not it exhibits aggregate uncertainty.

We close this section with two lemmas.

**Lemma 3.7:** It is weakly dominated to set \(b_{ih}(v) > v_{ih}\).

**Lemma 3.8:** There is \(\omega > 0\) and \(n^* < \infty\) such that \(E(f^n) > \omega E(s^n)\) for \(n > n^*\).

That is, expected feasible surplus grows as the expected number of active players. Lemma 3.8 follows from A4 (no asymptotic atoms), A5 (relevance), and the assumption that \(E(k^n) > \tau n\).

4. Efficiency of Discriminatory Auctions

We now make the link between AES and asymptotic efficiency. We begin with discriminatory auctions.

\(^{18}\) See McAfee and McMillan (1987) and Vleuguls (1997) for analyses of auctions with an unknown number of competitors.

\(^{19}\) This is automatic with asymptotically diffuse supply or with deterministic supply as long as \(k^n \to \infty\).
THEOREM 4.1: If \( \{A^n_i, e^n\} \) satisfies AES, then it is asymptotically ex-ante efficient. Under strong AES, the choice of \( e^n \) is immaterial.

So, under AES, discriminatory auctions are close to efficient for large \( n \), despite any asymmetries, multiple unit demands, or aggregate uncertainty. Under strong AES, this holds regardless of which equilibria are chosen. As we shall see, this means that under strong AES, equilibria are in the limit essentially unique.

Depending on the environment, the antecedents for either part of the theorem might be easier to check. For auctions that occur regularly it may be possible to form empirical estimates of the amount by which the actual \( P_i^n(\cdot) \) functions differ. For other auctions, observable exogenous features may be enough to guarantee strong AES.

4.1. Proof for the Case of Single Unit Demands

Pick \( \varepsilon > 0 \). We shall establish that for \( n \) large enough, the ex-ante expected efficiency loss on a per person basis is at most \( \varepsilon \). By Lemma 3.8, losses are thus small compared to \( Ef \).

Let \( 0 < \alpha < 1 \) be such that

\[
2\alpha\bar{v} + \alpha z\bar{v} + \alpha \leq \varepsilon .
\]

Let \( n^* \) be chosen so that for \( n > n^* \), and for all \( i, j, x, \) and \( b, \)

\[
|P_i^n(b) - P_j^n(b)| < \alpha^3/2\bar{v}^3,
\]

and

\[
\sum_{i=1}^n G_i^n(W(x, 2\alpha))/n < \alpha z .
\]

(4.2) is possible by AES, and (4.3) by A4 (no asymptotic atoms). By (4.2) for any \( b \in \bar{B}, v, \) and \( n > n^*, \)

\[
|S_i^n(v, b) - S_j^n(v, b)| = |P_i^n(b) - P_j^n(b)||v - b| \\
\leq |P_i^n(b) - P_j^n(b)|\bar{v} < \alpha^3/2\bar{v}^2 .
\]

Since \( S_i(v, \cdot) \) may be close to flat over some region, even small differences between \( S_i(v, \cdot) \) and \( S_j(v, \cdot) \) could make their maximizers far apart. On the other hand, if for a given \( v, S_i(v, b) > S_j(v, b'), \) then \( i \) will also prefer \( b \) to \( b' \). The next Lemma uses this observation to establish our key result: if \( j \) has value much below \( i \), then \( i \) does not bid much below \( j \).

Let \( \hat{u} = \min \{v : K_i^n(\hat{v}) \geq 2\alpha \text{ for some } i \} \), and let \( i^{*n} \) be a player for whom this minimum is achieved. Consider any \( n > n^* \), and player \( \hat{i} \) in \( A^n \). Through the remainder of this proof, we suppress the \( n \) superscript.
LEMMA 4.2: Choose \( v \geq \hat{v} + 2\alpha \), and \( v' < v - \alpha \). Let \( b_j \in BR_i(v) \), and \( b_j \in BR_j(v') \) for some \( j \). Then, \( b_j > b_j - \alpha \).

PROOF: By Lemma 3.7, \( P_i(\hat{v}) \geq K_i(\hat{v}) \geq 2\alpha \). Hence, \( P_i(\hat{v}) \geq 2\alpha - \alpha^3/2v^3 \geq \alpha \). So, for any \( \epsilon > 0 \), player \( i \) can bid \( \hat{v} + \epsilon \) and earn at least \( \alpha(v - \hat{v} - \epsilon) \geq \alpha(2\alpha - \epsilon) \). As this holds for any \( \epsilon > 0 \), it follows that \( S_i(v) \geq 2\alpha^2 \). Hence, since \( b_j \in BR_i(v) \), \( P_i(b_j)(v - b_j) = S_i(v) \geq 2\alpha^2 \), and so, since \( v - b_j \leq v \),

\[
(4.5) \quad P_i(b_j) \geq 2\alpha^2/v.
\]

If \( P_i(b_j) < \alpha^2/v \), then, since \( |P_i(b_j) - P_i(b_i)| < \alpha^2/2v^3 \leq \alpha^2/v \), it follows that \( b_i \leq b_j \) and we are done. So, assume \( P_i(b_j) \geq \alpha^2/v \), and let \( \hat{b} \leq b_j - \alpha \). Since \( b_j \in BR_i(v) \),

\[
(4.6) \quad S_j(v', b_j) - S_j(v', b) \geq 0
\]

and thus

\[
(4.7) \quad P_j(b) \leq P_j(b_j)(v' - b_j)/(v' - b).
\]

But then,

\[
S_j(v, b_j) - S_j(v, b) = S_j(v', b_j) + (v - v')P_j(b_j) - (S_j(v', b) + (v - v')P_j(b)) \\
> \alpha \left( P_i(b_j) - P_i(b) \right) \quad \text{(using 4.6, and since } v - v' > \alpha \text{)} \\
\geq \alpha P_i(b_j)(b_j - b)/(v' - b) \quad \text{(using 4.7)} \\
\geq P_j(b_j) \frac{\alpha}{v} \quad \text{(since } b_j - b \geq \alpha \text{ and } v' - b \leq v) \\
\geq \frac{\alpha^3}{v^2} \quad \text{(by assumption)}. \]

Therefore,

\[
S_i(v, b_j) - S_i(v, b) = S_i(v, b_j) - S_i(v, b_j) + S_j(v, b_j) - S_j(v, b) + S_j(v, b) - S_j(v, b) \\
> - \alpha^3/2v^2 + \alpha^3/v^3 - \alpha^3/2v^3 = 0.
\]

Thus for all \( b \leq b_j - \alpha \), \( S_i(v, b_j) - S_i(v, b) > 0 \) and so \( b \notin BR_i(v) \). Since \( b_j \in BR_i(v) \), it follows that \( b_i \geq b_j - \alpha \) and we are done. Q.E.D.

Let us consider the loss in surplus from \( i \) losing when he should win. If \( v_i < \hat{v} \) then, by definition of \( \hat{v} \), \( v_i \) is among the \( k \) th highest values at most \( 2\alpha \) of the...

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20 The \( \epsilon \) is to take account of the possibility that \( P_i(\cdot) \) has an atom at \( \hat{v} \). A player can always come arbitrarily close to his maximal payoff using only bids in \( \hat{B} \). We will henceforth behave as if \( \hat{B} = \hat{B} \), skipping tedious arguments in this style.
time. The loss in surplus from $i$ losing when he should have won with value $v_i \leq \hat{v}$ is thus at most $2\alpha \bar{v}$.

If $i$ always loses with $v_i \in [\hat{v}, \hat{v} + 2\alpha)$, and efficiently always should win, the loss in ex-ante surplus is at most $G_i^n(W(\hat{v}, 2\alpha))\bar{v}$.

Consider $v_i \geq \hat{v} + 2\alpha$. Assume first that no one who wins has value below $v_i - \alpha$. Then, if $i$ loses, he does so to someone with value at least $v_i - \alpha$. So, the efficiency loss from this event is at most $\alpha$.

Let $P^*$ be the probability that some player $j$ with value below $v_i - \alpha$ wins but $i$ does not win, given that $i$ bids $b_i \in BR_i(v_i)$. By Lemma 4.2, $b_i \leq b_i + \alpha$ for any such $j$, and so

$$P_i(b_i + \alpha) \geq P_i(b_i) + P^*.$$  

But, since $b_i \in BR_i(v_i)$,

$$P_i(b_i)(v_i - b_i) \geq P_i(b_i + \alpha)(v_i - b_i - \alpha)$$

$$\geq (P_i(b_i) + P^*)(v_i - b_i) - \alpha,$$

from which

$$P^*(v_i - b_i) \leq \alpha.$$  

But, $P^*$ was defined as the probability that $i$ loses but someone with value below $v_i - \alpha$ wins given $i$ bids $b_i$, while $v_i - b_i$ is an upper bound on the efficiency loss when this occurs (since whoever won an object paid at least $b_i$, and so by Lemma 3.7 places value at least $b_i$ on the object). So, the total loss from this event is bounded by $\alpha$. Since the two events are comprehensive but exclusive, the efficiency loss from $i$ losing with value $v > \hat{v} + 2\alpha$ is at most $\alpha$.

Summing across the events $v < \hat{v}, v \in [\hat{v}, \hat{v} + 2\alpha]$, and $v > \hat{v} + 2\alpha$, and noting that $i$ is only present $E^s(n)/n$ of the time, the ex-ante efficiency loss from $i$ losing when he should have won is at most

$$\left(E(s^n)/n\right)(2\alpha \bar{v} + G^n_i(W(\hat{v}, 2\alpha))\bar{v} + \alpha).$$

Summing across players, and using (4.3) and (4.1), we conclude that

$$E(a^n) \geq E(f^n) - \varepsilon E(s^n).$$

Thus, by Lemma 3.8,

$$E(a^n)/E(f^n) \geq 1 - \varepsilon E(s^n)/\omega E(s^n) = 1 - \varepsilon/\omega.$$  

Since $\varepsilon > 0$ was arbitrary, we are done.  

4.2. Proof for the Case of Multiple Unit Demands

In this section, we show how the argument of the previous section extends to the case of multi-unit demands.

A key part of the proof in the last section was Equation (4.5) establishing a minimum probability with which optimal bids win. An analogue in this setting is the following lemma.
LEMMA 4.3: Assume \( \{A^n_D, e^n\} \) satisfies AES. Choose \( 1 > \alpha > 0 \). Then, there is \( n^* < \infty \) such that for all \( n > n^* \), for all \( i \in V \), for all \( h \) such that \( v_{ih} \geq \hat{v}^n + \alpha \), and for all \( b_i \in BR_i(v_i) \),
\[
P_i^n(b^n_{ih}) \geq \alpha^{m+1}/2\pi^m.
\]

For \( m = 1 \), this follows directly from above. For \( m > 1 \), \( b_{ih} \) is constrained to be no greater than \( b_{i,h-1} \). So, the fact that \( S_{ih}(v_{ih}, b_{ih}) \) is very small even though \( v_{ih} > \hat{v}^n + \alpha \) is not evidence that \( b_i \) is suboptimal. The proof is inductive: if \( P^n_{i,h-1}(b^n_{i,h-1}) \) is not too small, then \( P^n_{i,h}(b^n_{i,h}) \) cannot be either.

The analogue to Lemma 4.2 in this setting is the following lemma.

LEMMA 4.4: Assume \( \{A^n_D, e^n\} \) satisfies AES. Then, for all \( 1 > \alpha > 0 \), there is \( n^* < \infty \) such that for all \( n > n^* \), for all \( j \neq i, h, h' \), and \( v_i, v_j \) such that \( v_{jh'} < v_{ih} - \alpha \) and \( v_{ih} > \hat{v}^n + \alpha \) and for all \( b_j \in BR_i^*(v_j) \), and \( b_i \in BR_i^*(v_i) \),
\[
b_{ih} > b_{jh'} - \alpha.
\]

The idea of the proof is very similar to that of Lemma 4.2. Assume that \( b_{ih} \leq b_{jh'} - \alpha \). Since with value \( v_{jh'} \), \( j \) weakly prefers \( b_{jh'} \) to \( b_{ih} \), with value \( v_{ih} \), \( j \) prefers \( b_{jh'} \) to \( b_{ih} - \alpha \) by an amount large enough to swamp the difference between \( i \) and \( j \)'s decision problems. As before, this uses that \( b_{ih} \) and \( b_{jh'} \) are at least \( \alpha \) apart, and that \( b_{jh'} \) wins with some minimum probability. The proof is complicated by the fact that since players submit \( m \)-vectors of bids, \( j \) may have other bids in the interval \( (b_{ih}, b_{jh'}) \).

Adding up the surplus losses and establishing that they grow small compared to feasible surplus as \( n \to \infty \) is as before.

5. EFFICIENCY OF UNIFORM PRICE AUCTIONS

We now turn to uniform price auctions. For discriminatory auctions, asymptotic efficiency holds both with no limit uncertainty (see Swinkels (1999)) and under AES (this paper). One might thus expect that as uniform price auctions become large, they too become efficient whether or not AES is satisfied. We begin with an example showing that mere largeness is not enough.

5.1. An Example of Limit Inefficiency

Consider a \( k + 1 \)st price auction. Assume there are \( n \) players each wanting two objects, and \( k = n \) objects. For each \( i \), take two independent uniform draws from \([2, 3]\). The higher draw is \( v_{i1} \) and the lower \( v_{i2} \). In any equilibrium satisfying weak dominance, \( b_{i1} = v_{i1} \). We claim that it is an equilibrium to have \( b_{i2} = 0 \). To see this, note that given this behavior, the \( k \)th highest bid is at least 2, while the \( k + 1 \)st is 0. So, each player gets surplus \( v_{i1} \). Setting \( b_{i2} \in (0, 2) \) raises the price but does not win a second object. Setting \( b_{i2} \geq 2 \) earns \( v_{i1} + v_{i2} - 2b_{i2} \leq v_{i1} + 3 - 2 \cdot 2 = v_{i1} - 1 \), and so is again strictly costly.
The surplus generated by the equilibrium is $E(v_{ij}) = 2.67$ per player (this is the expectation of the higher of two independent draws from the uniform distribution on $[2, 3]$). On the other hand, in an efficient allocation, any unit of demand with valuation above 2.5 is filled with probability going to 1, while values below 2.5 are filled with probability going to 0. Since each player’s values are derived from two independent uniform draws on $[2, 3]$, expected feasible surplus per person converges to 2.75. Hence, inefficiency persists no matter how large is $n$.

5.2. The Theorem

The key to the preceding example is that over the interval $(0, 2)$, raising one’s second bid increases the price but doesn’t change the allocation. The example thus fails AES violently: for each $b \in (0, 2)$, $P_i^n(b) = 1$ while $P_i^n(b) = 0$. And indeed, while mere largeness is not enough for asymptotic efficiency, AES is:

**THEOREM 5.1:** If $(A^n_U, e^n)$ satisfies AES then it is asymptotically ex-ante efficient. Under strong AES, the choice of $e^n$ is immaterial.

The idea behind Theorem 5.1 is that under AES, a player’s behavior has vanishing effect on the $k$ or $k + 1$st order statistics on bids. So, even a very small chance of price $P < v_{ih}$ for an unfilled unit of demand makes raising $b_{ih}$ worthwhile. Thus, in equilibrium, it is almost never the case that there are unfilled units with value much above $P$. Since all units won have value at least equal to $P$, the efficiency loss vanishes. Formally, we have the following proof.

**PROOF:** Fix $n$. For each $i$, let $S_i$ be (the random variable denoting) $i$’s equilibrium surplus in $A$. Let $H_i$ be the number of objects $i$ wins, and let $P$ be the price. Then,

$$S_i = \sum_{1 \leq h \leq H_i} v_{ih} - H_i P. $$

Let $S_i', H_i', P_i'$ be analogously defined assuming that $i$ sets $b = v$. Then,

$$S_i' = \sum_{1 \leq h \leq H_i'} v_{ih} - H_i' P_i'. $$

Since bids are weakly less than values, $H_i' \geq H_i$ and $P_i' \geq P$. Thus

$$S_i' - S_i = \sum_{H_i < h \leq H_i'} (v_{ih} - P) - H_i'(P_i' - P).$$

Some extra units are won, but the price is increased.

Let $r_i$ be the $k^n + 1$st highest bid by players other than $i$, and $r_i'$ the $k^n - m$th. Then, even if $i$ submits $m$ bids of 0, the $k^n + 1$st bid is at least $r_i$, and hence so is $P$. Conversely, even if $i$ submits $m$ bids of $\bar{v}$, the $k^n$th bid is at most $r_i'$, and hence so is $P$. Thus, $r_i' \geq P_i' > P \geq r_i$. 


Since $S$ is the surplus from optimal play, $E(S' - S_i) \leq 0$. Hence

$$E\left( \sum_{H_i < h \leq H'_i} (v_{ih} - P) \right) \leq E(H'_i(P'_i - P)) \leq m(E(r'_i) - E(r_i)).$$

By definition, $P_{i0}^n(\cdot)$ is the cumulative for $r_i$, and $P_{i,m+1}^n(\cdot)$ for $r'_i$. So,

$$E(r'_i) = \int_0^P xdP_{i,m+1}^n(x) = \int_0^P (1 - P_{i,m+1}^n(x)) \, dx$$

and similarly for $E(r_i)$. Thus,

$$m(E(r'_i) - E(r_i)) = m \int_0^P (P_{i0}^n(x) - P_{i,m+1}^n(x)) \, dx.$$

By AES, for any $\epsilon$, there is $n^*\epsilon$ such that for all $x$, and $n \geq n^*$, $P_{i0}^n(x) - P_{i,m+1}^n(x) < \epsilon/m\tilde{v}$. Hence

$$E\left( \sum_{H_i < h \leq H'_i} (v_{ih} - P) \right) < \epsilon.$$

Let $J_i$ be the number of objects $i$ efficiently receives. The efficiency loss is then

$$f^n - a^n = \sum_{i \in X^n} \left( \sum_{1 \leq h \leq J_i} v_{ih} - \sum_{1 \leq h \leq H_i} v_{ih} \right)$$

$$= \sum_{i \in X^n} \left( \sum_{H_i < h \leq J_i} v_{ih} - \sum_{J_i < h \leq H_i} v_{ih} \right)$$

$$\leq \sum_{i \in X^n} \sum_{H_i < h \leq J_i} (v_{ih} - P)$$

using that $\sum_{i \in X^n}(J_i - H_i) = 0$ and that every filled unit of demand has value at least $P$. Now, $J_i \leq H'_i$, since $H'_i$ is the number of objects $i$ wins when he bids truthfully while other players follow equilibrium, and so at least weakly understate their valuations. Hence, the last expression in (5.2) is at most

$$\sum_{i \in X^n} \sum_{H_i < h \leq H'_i} (v_{ih} - P),$$

and so

$$E(f^n - a^n) \leq \sum_{i \in X^n} E\left( \sum_{H_i < h \leq H'_i} (v_{ih} - P) \right) \leq E(s^n)\epsilon.$$

The result follows from Lemma 3.8. \textit{Q.E.D.}

A correlate is that under strong AES, bidding honestly is “almost weakly dominant”: bidding honestly never costs much, and guarantees receiving as many objects as is profitable for any play by the opponents.
5.3. Another Route to AES

For the special case of $k + 1$st price auctions, there is an additional way to guarantee AES. In this setting, it is weakly dominant to set $b_i = v_{i1}$. So, enough noise in the various $v_{i1}$’s will ensure that for each $b$, there are many players whose highest bid may be either greater or less than $b$. A proof similar to that for idiosyncratic demand uncertainty then establishes AES. We omit a formal development.

6. LIMIT REVENUE EQUIVALENCE AND BEHAVIOR OF BID FUNCTIONS

For a single unit auction, the revenue equivalence result (see e.g., Milgrom and Weber (1982)) states that for any auction, any efficient equilibrium that yields the lowest value 0 surplus yields revenue equal to the expectation of the second highest value. In Weber (1983) and Engelbrecht-Wiggans (1988) this is generalized to show that with multiple units (but single unit demands) revenue per unit is the expectation of the $k + 1$st value. Along the lines of these results, in this section we use the asymptotic efficiency result to characterize limit revenue. This then allows us to characterize limit bid behavior.

For these results, we require that no interval contains a vanishing fraction of value realizations in the limit:

**Assumption (No Asymptotic Gaps):** There is a continuous function $M(\cdot)$ such that for all $\varepsilon > 0$, $M(\varepsilon) > 0$, and for all $x \in [\underline{v}, \overline{v} - \varepsilon]$ and $n$,

$$\sum_{i=1}^{n} \frac{G_{i}(W(x, \varepsilon))}{n} \geq M(\varepsilon).$$

Under no asymptotic gaps, $K_{ih}^n(\cdot)$ does not depend much on either $i$ or $h$.

**Lemma 6.1:** Assume $(A^n)$ satisfies no asymptotic gaps. Then, for all $\varepsilon > 0$, there is $n^* < \infty$ such that for all $n > n^*$, $i, j, h, h'$, and $x > \varepsilon$,

$$K_{ih}^n(x) \geq K_{h'j}^n(x - \varepsilon) - \varepsilon.$$

Intuitively, with no asymptotic gaps, value realizations, and hence their order statistics, are almost always packed close together.

To reduce the notational load in what follows, for each $n$, fix an arbitrary $i_n \in \{1, \ldots, n\}$, and $h_n \in \{1, \ldots, m\}$, and define $K^n(\cdot) = K_{i_n h_n}^n(\cdot)$, $P^n(\cdot) = P_{i_n h_n}^n(\cdot)$, and $y_n = y_{i_n}^n(k^n - h_n + 1)$. By our previous results, the behavior of these objects does not depend much on $i$ or $h$.

The following is the $m$ unit version of the insight underlying the standard revenue equivalence result.
LEMMA 6.2: \( S^n(v) \) is convex (and hence continuous and differentiable almost everywhere). At each \( v \in V \), and for each \( b \in BR^n(v) \) there is a supporting hyperplane to \( S^n(v) \) with gradient \( P^n_i(b) \).

PROOF: Since values are independent, neither \( P^n_i(v) \) nor expected payments with any given \( b \) depend on \( v \). So, if \( i \) bids \( b \) with any given value \( w \), he gets surplus

\[
S^n_i(v) + \sum_{h=1}^{m} P^n_i(b_h)(w_h - v_h).
\]

As this is a lower bound for \( S^n_i(w) \), we are done. \( Q.E.D. \)

The next lemma is the key to our asymptotic characterization of surplus. For \( \varepsilon > 0 \), define \( V_\varepsilon \) as the subset of \( V \) on which successive nonzero values are separated by at least \( \varepsilon \). That is,

\[
V_\varepsilon = \{ v \in V | v_h > 0 \Rightarrow v_h - v_{h+1} > \varepsilon \ \forall h = 1, \ldots, m-1 \}.
\]

Because of no asymptotic atoms, values almost always fall in \( V_\varepsilon \) for \( \varepsilon \) small.

LEMMA 6.3: If \( \{ A^n, e^n \} \) satisfies AES, then for all \( \varepsilon > 0 \), there is \( n^* < \infty \) such that for all \( n > n^* \), \( i, \ v \in V_\varepsilon \), \( b \in BR^n(v) \), and \( h \),

\[
(6.1) \quad K^n(v_h - \varepsilon) - \varepsilon < P^n_i(b_h) < K^n(v_h + \varepsilon) + \varepsilon.
\]

So, the probability with which one wins in equilibrium must approximate the efficient probability of winning. This is intuitively obvious given that AES implies asymptotic efficiency. Using no asymptotic gaps, we show that a failure of (6.1) at a single value implies a misallocation on a positive measure set of values, contradicting asymptotic efficiency.

By Lemma 6.2, the slope of the surplus function is equal to the probability of winning in equilibrium. By Lemma 6.3, this probability converges to \( K^n(v_h) \) on any given unit. Integration by parts thus gives the following theorem.

THEOREM 6.4: If \( \{ A^n, e^n \} \) satisfies AES, then uniformly over all \( i \) and \( v \),

\[
(6.2) \quad S^n_i(v) \to \sum_{h=1}^{m} K^n(v_h)[v_h - E(y^n | y^n < v_h)].
\]

So, expected surplus on \( v_h \) approaches the probability that \( i \) efficiently wins an \( h \)th object multiplied by the expectation of the difference between \( v_h \) and the marginal value in an efficient allocation.

Finally, since we have characterized both the asymptotic surplus and probability of winning, we have the following corollary.
COROLLARY 6.5: If \( \{A^n_i, e^n\} \) satisfies AES, then, for all \( \varepsilon > 0 \), uniformly over all \( i, h, \) and \( v_i \) such that \( K^n(v_{ih}) > \varepsilon \),

\[
b^n_{ih}(v_i) \to E(y^n | y^n < v_{ih}).
\]

Intuitively, since players win with probabilities converging to \( K^n(v_{ih}) \), bids must converge to \( E(y^n | y^n < v_{ih}) \) to satisfy (6.2). So, for any \( v_{ih} \) that has a nontrivial probability of being efficiently filled, \( b_{ih} \) converges to the expectation of \( y^n \) conditional on \( y^n < v_{ih} \). This is the natural generalization of the symmetric equilibrium of a standard first price auction, which has each player bid the expectation of the second highest valuation conditional on that valuation being less than his own. Intuitively, since players win with probabilities converging to \( K^n(v_{ih}) \), bids must converge to \( E(y^n | y^n < v_{ih}) \) to satisfy (6.2).

One might conjecture that in uniform price auctions, bids would also be pushed close to value. To see that this need not be so, reconsider the example from Section 5.1 with supply uniformly distributed on the integers in \( (n - \sqrt{n}, n + \sqrt{n}) \). Then, \( \{A^n\} \) satisfies strong AES, and so by Theorem 5.1, any \( \{e^n\} \) is asymptotically efficient. \( K^n(\cdot) \) converges to a step function at 2.5, and so from Theorem 6.4, \( S_{ih}(v_{ih}) \to \max(0, v_{ih} - 2.5) \). Hence, \( p \to 2.5 \) as well. However, we cannot rule out that, for example, \( b^n_{ih}(v) = 2.6 \) when \( v_{ih} = 2.9 \), because this distortion is in the limit irrelevant to both the allocation and the price.

### 6.1. Full Support Uncertainty and Uniform Price Auctions

A tighter characterization of equilibrium play in uniform price auctions requires an additional assumption that the market clearing price in the analog competitive market continues to fall into any interval of \( [0, \bar{v}] \) with positive probability. Because of law of large numbers considerations, this is stronger than no asymptotic gaps.

**Definition 6.6:** \( \{A^n\} \) has full support uncertainty if for all \( \varepsilon > 0 \), there is \( \delta > 0 \) such that \( K^n(x) - K^n(x - \varepsilon) > \delta \) for all \( n \) and \( x \in (\varepsilon, \bar{v}) \).

**Corollary 6.7:** If \( \{A^n_i, e^n\} \) satisfies AES and full support uncertainty, then \( b_i \to v_i \), uniformly over \( i \) and \( v \).

We dispense with a formal proof. The idea is as follows. Assume that \( b_i < v_i - \varepsilon \). Then, by (5.1), it must be that \( P^n(v_i - \varepsilon/3) - P^n(v_i - \varepsilon) \) is very small, else \( b = v \) gives a nontrivial probability of winning an extra object at a price \( p \leq v - \varepsilon/3 \). Since \( K^n(\cdot) \) is strictly increasing, and \( K^n(v - \varepsilon/3) \leq P^n(v - \varepsilon/3) \) (by Lemma 3.7), \( K^n(\cdot) \) and \( P^n(\cdot) \) must thus differ by a nonvanishing amount over \( (v_i - \varepsilon, v_i - 2\varepsilon/3) \), contradicting our asymptotic characterization of revenue.
6.2. Individual AES

Our results thus far have depended on full equilibrium reasoning and the assumption of common priors. That both are standard does not make them more attractive. For the discriminatory auction case, it seems unlikely that any substantial weakening beyond $\epsilon$-equilibria (see below) is possible, since it is critical to that case that players’ beliefs about their strategic environment are in the limit identical. Consider a setting in which $i$ believes supply is likely to be quite small, or that lots of people are likely to show up at the auction, while $j$ believes the opposite. Then, $i$ will shade his bids less than $j$, and this is not mitigated by the presence of many opponents. This is why we required all subsets of size $s^n$ to be equally likely in defining aggregate demand uncertainty.

For the uniform price case, things are different: under individual AES, (5.1) will hold from the point of view of each $i$ even if players have different beliefs about demand and supply, and even if beliefs are incorrect or out of equilibrium. Of course, if beliefs are incorrect, then this need not imply efficiency as viewed from an outside observer. However, if each $i$ has beliefs about price that satisfy a full support assumption similar to Definition 6.6, then each $i$ will optimally bid close to value and the actual outcome will be near efficient regardless of whether beliefs are consistent with reality.

7. $\epsilon$-Equilibria

Say that a set of strategies is an $\epsilon$-equilibrium if for each $i$ and $v_i$, $i$ earns within $\epsilon$ of the best feasible payoff.

**Theorem 7.1:** Choose $\epsilon > 0$. For each $n$, there is an $\epsilon$-equilibrium of $A^n$. If $\{A^n\}$ satisfies strong AES, then for $n$ sufficiently large, $A^n$ has a symmetric $\epsilon$-equilibrium. For $\{A^n_i\}$, one symmetric $\epsilon$-equilibrium has $b^n(v_i) = E(y^n|y^n < v_i)$ for each $v$ and $h$ at all points of continuity of $K^n(\cdot)$. For $\{A^n_i\}$, one symmetric $\epsilon$-equilibrium is for each player to set $b = v$.

The existence of $\epsilon$-equilibria is standard, based on finite grids of bids and types. For the discriminatory case, if players bid according to $b^n(v_i) = E(y^n|y^n < v_i)$, then $P_{i, k}(b^n(v_i)) \approx P(y^n < v_i)$. But then $b^n(v_i)$ is a best response (since it is the solution to the standard auction). We show that this can be extended to an $\epsilon$-equilibrium in which players symmetrically randomize at atoms in $K^n(\cdot)$. For the uniform case, the result is obvious since under strong AES, bidding honestly has vanishing cost.

So, $\epsilon$-equilibria exist. And, the bidding functions identified are symmetric, strictly increasing in $v_i$, and independent of $h$, and thus efficient. However, this does not rule out that other $\epsilon$-equilibria might be substantially inefficient.

**Corollary 7.2:** Assume $\{A^n\}$ satisfies strong AES. Then, for each $\delta > 0$, there is $\epsilon > 0$ such that for any sequence of $\epsilon$-equilibria $\{A^n\}$, $\lim E(a^n)/E(f^n) > 1 - \delta$. 
So, when players play close to optimally, then asymptotically the auction is close to efficient.

We sketch the proof. For the discriminatory case, the key was Lemma 4.2 (or for $m > 1$, Lemma 4.4). The idea is that for large enough $n$, if $b_j$ is a best response for $j$ with value $v' < v - \alpha$, then with $v$, $j$ prefers $b_j$ to $b_j - \alpha$ by enough to take care of the approximation involved in comparing $i$ and $j$'s payoffs. This extends directly to $\varepsilon$-equilibria: for $\varepsilon$ small relative to $\alpha$ and for $n$ large, if $b_j$ is an $\varepsilon$-best response for $j$ with $v'$, then with $v$, $j$ prefers $b_j$ to $b_j - \alpha$ by an amount large enough to cover the approximation in moving from $j$ to $i$, and to make $b_j - \alpha$ suboptimal for $i$ by more than $\varepsilon$. The only other places where optimization is used in the proof are in establishing that when $v_i > \hat{v} + \alpha$, $b_j$ wins with some minimal probability, and that players do not bid above their valuations. For each of these, an $\varepsilon$ version is obvious.

For the uniform case, the result follows since honest bidding has vanishing cost in terms of raising the price, and since the gains from deviating in terms of winning additional objects overstate the efficiency loss.

8. CONCLUSION

We have shown that under AES, private value auctions converge to efficient outcomes even in a setting with aggregate uncertainty, asymmetric players, and multiple unit demands. Limit bidding behaviors accord well with what we have learned in simpler settings.

The arguments for the discriminatory and uniform price cases differ in their structure (which is simpler for the uniform case than for the discriminatory) and more importantly, in the strength of assumptions needed. In particular, the assumption of common priors is critical in the discriminatory setting, but not in the uniform. This reflects the different forms of reasoning in the simplest single unit case. For discriminatory auctions the full force of equilibrium reasoning is needed, and symmetry is critical. Not surprisingly then, the asymptotic characterization also depends on fairly full-fledged equilibrium reasoning, and on a form of limit symmetry among bidders: In the limit, bidders have to believe the same things about their environments. To have this limit symmetry in beliefs, one needs, among other things, common priors. In the second price case, a simple weak dominance argument ties down optimal behavior even out of equilibrium, and regardless of symmetry. The argument for efficiency in the limit of these auctions is also essentially one of limit weak dominance. It is enough that each player believes he has a small effect on his environment; whether players have the same beliefs is irrelevant.

A number of papers (see, e.g., Milgrom and Roberts (1994), Milgrom and Shannon (1994), and Topkis (1978)) discuss conditions under which parameterized decision problems have “monotone comparative statics.” In those settings, if an increase in a decision variable is desirable for one type, then it is strictly
desirable for higher types. The driving force behind efficiency for the discriminatory case is a form of “monotone comparative statics with strict slope”: When j’s value increases by α, an α increase in bid that used to be weakly desirable is now desirable by an amount bounded away from zero. This allows us to draw conclusions about i’s best responses. It would be interesting to know which other parameterized decision problems exhibit this structure. Since the condition is cardinal, it seems unlikely that the order theoretic techniques discussed in the references above will be enough.

Rustichini, Satterthwaite, and Williams (1994) consider asymptotic efficiency for large double auctions. In their setup, all buyers (respectively, sellers) have IID values for a single unit, and follow symmetric equilibrium behavior. Inefficiency arises from sellers overstating, and buyers understating, valuations, so that the equilibrium outcome involves fewer transactions than is optimal. Without the symmetry and single unit demand assumptions some objects may also end up being sold by the wrong seller or to the wrong buyer. The extension of our techniques to double auctions is left for future research.

Also in contrast to Rustichini et al., this paper lacks a rate of convergence result. Their symmetric single unit demand and supply setup allows an approach based on first order conditions. Our more general setup does not. One suspects that a “fast convergence” result would rest on more structure than the current paper. As numerical techniques advance, it would be interesting to check the rate of convergence to efficiency in examples.

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**APPENDIX**

**Proof of Lemma 2.2:** By Swinkels (1999, Lemma 3.1), a sufficient condition for ex-ante efficiency to imply ex-post efficiency is that for all ε > 0, there is δ > 0 such that for n large enough, Pr(f^n > δE(f^n)) > 1 − ε. Let γ = (1 − ε)^(1/3). Choose n’ and δ such that for n > n’, Pr(s^n > δE(s^n)) > γ and Pr(k^n > δE(k^n)) > γ. Since E(k^n) ≥ τn ∀n, Pr(k^n > δτn) > γ. If m^n ≤ k^n, then regardless of equilibrium behavior, all units of demand are filled, and a^n/k^n = 1. Assume m^n > k^n and k^n > δτn. The expected surplus from randomly allocating a single unit of supply among the players in A^n is at least (1 − Q)/2z for all n sufficiently large (see the proof of Lemma 3.8). So, the expected surplus from randomly allocating a single object to k^n/m of the players in A^n is (k^n/m)(1 − Q)/2z = (δτn/m)(1 − Q)/2z. Since value vectors are drawn independently, there is n” > n’ such that for n > n”, and for s^n > k^n/m > δτn, Pr(E(f^n|s^n) < (δτn/m)(1 − Q)/2z) < γ. So for n > n”, at least (1 − γ)^3 of the time, f^n > (δ/2m)(1 − Q)/2z. Since E(f^n) ≤ nmτ, we are done by taking δ’ = (1/nmτ)(δτn/2m)(1 − Q)/2z = δτ(1 − Q)/4m^2τz.

**Q.E.D.**

**Proof of Theorem 3.6:** We prove the three cases separately.

**Lemma 9.1:** If {A^n} has asymptotically diffuse supply, then it satisfies AES.
PROOF: For all $M \subseteq \{1, \ldots, n\}$, and for all $b$,
\[
\Pr(k^n - 2m \leq B^*_b(b) \leq k^n) = \sum_{k=0}^{nm-1} \mu^0(k) \Pr(k - 2m \leq B^*_b(b) \leq k) \\
\leq \sum_{k=0}^{nm-1} 2m \Pr(B^*_b(b) = k - j) \\
= \sum_{j=0}^{2m-nm+1} \sum_{k=0}^{nm-1} \Pr(B^*_b(b) = k - j) \\
< (2m + 1) \Pr^n,
\]

since \( \sum_{k=0}^{m} \Pr(B^*_b(b) = k) = 1 \). By Lemma 3.2, we are done. Q.E.D.

**Lemma 9.2:** Let \( \{A^v\} \) have idiosyncratic demand uncertainty. Then, for all \( \epsilon > 0 \), there is \( k^* < \infty \) such that for all \( k > k^* \), all \( n \), all bidding behaviors, all \( M \subseteq \{1, \ldots, n\} \), and all \( b \in (0, \pi] \), \( \Pr(B^*_b(b) = k) < \epsilon \).

**Proof:** We shall establish that for all \( \epsilon > 0 \), there is \( k^* < \infty \) such that for all \( k > k^* \), \( n \), \( M \), and \( X^n \), \( \Pr(B^*_b(b-k) = k|X^n) < \epsilon \). Since this holds for all \( X^n \), the result follows. So, fix \( n \) and \( X^n \), and let \( \hat{M} = M \cap X^n \). For \( j \in \hat{M} \), let \( \delta_j \geq \delta \) be the minimum probability with which \( j \) submits all bids equal to 0. For given \( k \), let us solve for the behavior by players in \( \hat{M} \) that maximizes \( \Pr(B^*_b(b-k) = k) \) subject to the \( \delta_j \)'s. Note that replacing \( \delta_j \) by \( \tilde{\delta} \) for all \( j \) increases the set of allowable behaviors and so weakly increases \( \max \Pr(B^*_b(b) = k) \). Pick an arbitrary \( j \in \hat{M} \). Then,
\[
\Pr(B^*_b(b) = k) = \sum_{k=0}^{m} \Pr(B^*_b(b) = k) \Pr(M \setminus J(b) = k - \tilde{k})
\]
since \( B^*_b(b) \) and \( B^*_b(b) \setminus J(b) \) are independent conditional on \( X^n \). So, \( \Pr(B^*_b(b) = k) \) is maximized by setting \( B^*_b(b) = \tilde{k} \) as large as possible for some \( \tilde{k} \in \{0, 1, \ldots, m\} \) that maximizes \( \Pr(B^*_b(b) = k - \tilde{k}) \). Thus, among the bidding behaviors that maximize \( \Pr(B^*_b(b) = k) \) are behaviors in which each player \( j \) makes \( m \) bids of zero with probability \( \delta_j \), and with probability \( 1 - \delta_j \), makes \( \hat{m}_j \) bids greater than \( b \) and \( m - \hat{m}_j \) bids of 0, for some \( 0 \leq \hat{m}_j \leq m \). Fix such a bidding behavior, and let \( \hat{M}_h = \{j \in \hat{M} | \hat{m}_j = h\} \). If for all \( h \in \{1, \ldots, m\}, |\hat{M}_h| < k/m^2 \), then \( \Pr(B^*_b(b) = k) = 1 \). So, assume that, for some \( h^* \in \{1, \ldots, m\}, |\hat{M}_h| \geq k/m^2 \). Then,
\[
\Pr(B^*_b(b) = k) = \sum_{j=0}^{m} \Pr(B^*_b(b) = jh^*) \Pr(M \setminus J(b) = k - jh^*)
\]
again using independence conditional on \( X^n \). But this in turn is at most
\[
\max_{j=0, \ldots, |\hat{M}_h|} \Pr(B^*_b(b) = jh^*) = \max_{j=0, \ldots, |\hat{M}_h|} \Pr(w = j)
\]
where \( w \) is the number of members of \( \hat{M}_h \) who are not constrained to bid 0 conditional on \( X^n \). But,
\[
\frac{\Pr(w = j + 1)}{\Pr(w = j)} = \left( \frac{|\hat{M}_h|}{j + 1} \right) \left( 1 - \delta \right)^{|\hat{M}_h| - j - 1}
\]
\[
= \left( \frac{|\hat{M}_h|}{j + 1} \right) \left( 1 - \delta \right)^{|\hat{M}_h| - j - 1}
\]
\[
= \left( \frac{|\hat{M}_h| - j - 1}{j + 1} \right) \frac{1 - \delta}{\delta}.
\]
So, \( \Pr(w = j) \) is maximized for \( j^* \approx (1 - \delta) |M_0| - \delta \), where the approximation is arbitrarily good as \( |M_0| \to \infty \). Pick a positive integer \( J \). Then, since \( |M_0| \geq k/m^2 \), and since \( j^* \) is (approximately) proportional to \( |M_0| \), if \( k \) is large enough,

\[
\left( \frac{1}{2} \right)^{\frac{1}{2} \frac{|M_0|}{j+1} \frac{1 - \delta}{\delta}} \left( \frac{1}{2} \right)^{-\frac{1}{2}}
\]

for all \( j \in \{j^* - J, \ldots, j^* + J\} \). But then \( \Pr(w = j) \geq \frac{1}{j+1} \Pr(w = j^*) \) for all \( j \in \{j^* - J, \ldots, j^* + J\} \) and so in particular \( \Pr(w = j^*) \leq (1/J) \). Since \( J \) is arbitrary, as \( k \to \infty \), \( \Pr(w = j^*) \to 0 \). \( \text{Q.E.D.} \)

**Corollary 9.3:** Assume \( \{A^u\} \) has idiosyncratic demand uncertainty and that for all \( K > 0 \), \( \lim_{n \to \infty} \sum_{k=1}^{K} \mu^u(k) = 0 \). Then, \( \{A^u\} \) satisfies AES.

**Proof:** Choose \( \varepsilon > 0 \). Using Lemma 9.2, let \( k^* \) be such that for all \( k > k^* - 2m \), \( \Pr(B^u(b) = k) < \varepsilon/4m \), and let \( n^* \) be such that for all \( n > n^* \), \( \sum_{k=1}^{n} \mu^u(k) < \varepsilon/2 \). Then, for any \( n > n^* \),

\[
\Pr(k^* - 2m \leq B^u(b) \leq k^*) = \sum_{k=1}^{n} \mu^u(k) \Pr(k^* - 2m \leq B^u(b) \leq k^*) \\
\leq \frac{\varepsilon}{2} + \sum_{k > k^*} \mu^u(k) \sum_{h=0}^{2m} \Pr(B^u(b) = k - h) \\
\leq \frac{\varepsilon}{2} + \sum_{k > k^*} \mu^u(k) (2m + 1) \varepsilon/4m \leq \varepsilon.
\]

The result follows by Lemma 3.2. \( \text{Q.E.D.} \)

**Lemma 9.4:** Assume demand is asymptotically diffuse. Then, for all \( \varepsilon > 0 \) there is \( k^* < \infty \), \( n^* < \infty \) such that if \( n > n^* \), \( k > k^* \), then for all \( M \subseteq \{1, \ldots, n\} \) and \( b \in (0, \tau) \), \( \Pr(B^u(b) = k) < \varepsilon \).

The idea of the proof is as follows. For given behavior by the players in \( M \), let \( \xi = E(B^u(b)) | i \in M \), \( i \in X^u \) be the expected number of bids above \( b \) by a typical member of \( M \) if he is active. Then, \( E(B^u(b)) s^u - s = \xi |M| s/n \), since the expected number of players in \( M \) to show up given \( s^u = s \) is \( |M| s/n \). So if \( s^u \) differs substantially from \( n \xi/(|M| \xi) \), \( E(B^u(b)) | s^u = s \) will be substantially different from \( k \), and it will be unlikely that \( B^u(b) = k \). Since \( s^u \) is diffuse, the result follows. Details are available from the author.

Much as before, Lemma 9.4 implies the following corollary.

**Corollary 9.5:** Assume \( \{A^u\} \) has aggregate demand uncertainty and that for all \( K > 0 \), \( \lim_{n \to \infty} \sum_{k=1}^{K} \mu^u(k) = 0 \). Then, \( \{A^u\} \) satisfies AES.

This completes the proof of Theorem 3.6. \( \text{Q.E.D.} \)

**Proof of Lemma 3.7:** Assume player \( i \) with value vector \( v_i \) submits a bid set \( b_i \) such that \( b_{ih} > v_{ih} \) for some \( h \). Let \( H \) be the largest \( h \) for which \( b_{ih} > v_{ih} \), and consider the bid set \( b^*_i = \{b_{ij_1}, \ldots, b_{ih_{-1}}, v_{ih}, b_{iH+1}, \ldots, b_{im}\} \).

**Discriminatory Case:** If \( i \) wins less than \( H \) objects with \( b_i \), then \( b^*_i \) wins the same number of objects at the same total cost. If \( i \) wins \( H \) or more objects with \( b_i \), then either \( b^*_i \) wins the same number of objects at a strictly lower cost, or \( b^*_i \) wins one fewer unit, but since \( b_{iH} > v_{ih} \) and \( v_{ih} \) is nonincreasing in \( h \), this is also a strict improvement.

**Uniform Price Case:** Substituting \( v^*_i(H) \) for \( b^*_i(H) \) either leaves the price the same or reduces it (since price is weakly increasing in the vector of submitted bids). So, if \( i \) wins the same number of
objects as before, then he is weakly better off. The other possibility is that \( i \) wins one fewer object. In that case, it must be that for the original bid set the \( k \)th order statistics on bids was at most \( b_{ih} \) (since \( b_{ih} \) won), while the \( k \)th + 1st order statistic on bids was at least \( v_{ih} \) (since when \( i \) lowers his bid to \( v_{ih} \), he wins one fewer object). But then, price for the original bid set was at least \( v_{ih} \), and was strictly greater than \( v_{ih} \) unless there used to be a tie at \( v_{ih} \). So, by lowering his bid, \( i \) pays no more on units 1, \ldots, \( H - 1 \), but no longer wins an object for which he was paying at least value.

Q.E.D.

**Proof of Lemma 3.8:** Note that

\[
E(f^n) \geq E\left( \min(k^n/s^n, m)s^n \sum_{ih} v_{ih}/nm \right).
\]

since this is less than the surplus from randomly allocating whatever supply is realized among the players who show up. Since \( k^n \leq nm \) with probability 1, \( \min(k^n/s^n, m) \geq k^n/n \), and so since \( E(k^n/n) > \tau, E(f^n) \geq \tau E\left( E_{ih} v_{ih}/nm \right) \). But, since \( \sum G(0)/n < Q \), and since for all \( y \), \( G^\tau(0, y) \leq \tau \) for \( n \) large enough,

\[
E\left( \sum_{ih} v_{ih}/nm \right) \geq E\left( \sum_{i} v_{ii}/nm \right) \geq (1 - Q)/2m.
\]

To see this, note that to minimize \( E(\sum_i v_{ii}/nm) \) subject to A4 and A5, the best we can do is put weight \( Q \) on \( v = 0 \), and pack the remaining probability uniformly in the interval \([0, (1 - Q)/2] \). We are done by taking \( \omega = \tau(1 - Q)/2m \).

Q.E.D.

**Proof of Lemma 4.3:** For each \( n \), let \( \delta^n = \sup_{b_h}(P^n(b) - P^n_{ih}(b)) \). By AES, \( \delta^n \to 0 \). Fix \( n, v^n_i \), and \( b^n_i \in BR^n_i(v^n_i) \). By Lemma 3.7, \( P^n_i, (\hat{\cdot})^n \geq a^n \), and hence for \( n \) large enough that \( \delta^n < a^n \). \( P^n_i(\hat{\cdot})^n \geq a^n \) for all \( i \). So, if \( v_{ih}^n \geq \hat{\cdot}^n + a^n \) and \( b_{ih}^n < \hat{\cdot}^n \), then \( i \) can change \( b_{ih}^n \) to \( \hat{\cdot}^n \), and earn at least \( a^n \) on his first unit, without changing his earnings on bids \( 2, \ldots, m \). So, if \( b_{ih}^n \leq \hat{\cdot}^n \), then \( P^n_i(b_{ih}^n) \geq a^n \). By induction, assume \( P^n_i(b_{ih-1}^n) \geq a^n/\tau^{h-1} - (h - 1)\delta^n \), for some \( h \in \{1, \ldots, m\} \). Assume \( v_{ih}^n \geq \hat{\cdot}^n + a^n \). If \( b_{ih-1}^n \leq \hat{\cdot}^n \), then \( i \) can increase \( b_{ih}^n \) to \( \hat{\cdot}^n \) and earn at least \( a^n \) on his \( h \)th unit, without changing his earnings on bids other than the \( h \)th, and so it must be that \( P^n_i(b_{ih}^n) \geq a^n/\tau^{h-1} - (h - 1)\delta^n \). On the other hand, \( b_{ih-1}^n < \hat{\cdot}^n \), then since \( P^n_i(b_{ih-1}^n) \geq a^n/\tau^{h-1} - (h - 1)\delta^n \), \( P^n_i(b_{ih}^n) \geq a^n/\tau^{h-1} - h\delta^n \). So, by setting \( b_{ih}^n = b_{ih-1}^n \), \( i \) earns at least \( a^n/\tau^{h-1} - h\delta^n \) on his \( h \)th unit, again without changing his winnings on other units. So,

\[
P^n_i(b_{ih}^n) \geq a^n/\tau^{h-1} - h\delta^n.
\]

Thus, by induction, \( P^n_i(b_{ih}^n) \geq a^{m+1}/\tau^m - m\delta^n \) for all \( h \) such that \( v_{ih} > \hat{\cdot}^n + a^n \). The result follows since, by AES, \( \delta^n \to 0 \).

Q.E.D.

**Proof of Lemma 4.4:** Define \( S^n_i(v_i, b_h) = P^n_i(b_h) - P^n_{ih}(b_h) \), so that \( S^n_i(v_i, b) = \sum_{h=1}^m S^n_{ih}(v_i, b_h) \). Let \( n^* \) be chosen so that for \( n > n^* \), \( i, j, h, h', x \),

\[
|P^n_{ih}(x) - P^n_{ih}(x)| < \delta = \frac{\alpha^{m+3}}{4(m + 2)\tau^{m+2}}.
\]

and such that for all \( v_i \in V \), such that \( v_{ih} \geq \hat{\cdot}^n + a^n \), and for all \( b_h \in BR_i(v_i) \),

\[
P^n_{ih}(b_{ih}) \geq \frac{\alpha^{m+1}}{2\tau^m}.
\]
Then, \( P_{jk}(b_{jk}) \geq (a^{m+1}/2 \pi^m) - \delta > (a^{m+1}/4 \pi^m) \). So if \( P_{jk}(b_{jk}) < (a^{m+1}/4 \pi^m) \), then \( b_{jk} < b_{jh} \), and we are done. Assume

\[
(9.3) \quad P_{jk}(b_{jk}) > \frac{a^{m+1}}{4 \pi^m}
\]

and \( b_{jk} - \alpha > b_{jh} \). We will show this leads to a contradiction.

Consider lowering \( j \)'s bid of \( b_{jk} \) to \( b_{jh} \), keeping other bids the same. Let \( T \geq 0 \) be the largest integer such that \( b_{jh} < b_{jh+T} \). Then, \( j \) is effectively lowering \( b_{jh} \) to \( b_{jh+1}, b_{jh+1} \) to \( b_{jh+2}, \ldots \), and \( b_{jh+T} \) to \( b_{j0} \). Since \( b_j \in BR(v_j) \) this change does not increase \( j \)'s payoff. So,

\[
0 \leq S_{jk}(v_{jk}, b_{jk}) - S_{jk}(v_{jk}, b_{jh+1}) + S_{jk}(v_{jh+1}, b_{jh+1}) - S_{jk}(v_{jh+1}, b_{jh+2}) + \cdots + S_{jk}(v_{jh+T}, b_{jh+T}) - S_{jk}(v_{jh+T}, b_{jh}).
\]

Note for each \( t = 0, \ldots, T \), since \( v_{jh+1} < v_{jh} = \alpha \),

\[
S_{jh+1}(v_{jh+1}, b_{jh+1}) - S_{jh+1}(v_{jh+1}, b_{jh+2}) \leq S_{jh+1}(v_{jh} - \alpha, b_{jh+1}) - S_{jh+1}(v_{jh} - \alpha, b_{jh+2}),
\]

where \( b_{jh+T} = b_{jh} \). To see this, note that

\[
S_{jh+1}(v_{jh+1}, b_{jh+1}) - S_{jh+1}(v_{jh+1}, b_{jh+2}) = P_{jh+1}(b_{jh+1}) - P_{jh+1}(b_{jh+2}) - P_{jh+2}(b_{jh+2}) + P_{jh+2}(b_{jh+1}).
\]

Since \( P_{jh+1}(b_{jh+1}) = P_{jh+1}(b_{jh+2}) \), this difference is increased when \( v_{jh+1} \) is replaced by \( v_{jh} - \alpha > v_{jh+1} \). So,

\[
0 \leq S_{jh}(v_{jh} - \alpha, b_{jh}) - S_{jh}(v_{jh} - \alpha, b_{jh+1}) + S_{jh+1}(v_{jh} - \alpha, b_{jh+1}) - S_{jh+1}(v_{jh} - \alpha, b_{jh+2}) + \cdots + S_{jh+T}(v_{jh} - \alpha, b_{jh+T}) - S_{jh+T}(v_{jh} - \alpha, b_{jh}).
\]

Now, for each \( t = 0, \ldots, T \),

\[
S_{jh+1}(v_{jh} - \alpha, b_{jh+1}) \geq S_{jh+1}(v_{jh} - \alpha, b_{jh+1+T}).
\]

This is so as \( v_{jh} > v_{jh+1+T} \geq b_{jh+1+T} \), where the second inequality uses Lemma 3.7 and \( P_{jh+1}(\cdot) \geq P_{jh+1+T}(\cdot) \). So,

\[
(9.4) \quad 0 \leq S_{jk}(v_{jh} - \alpha, b_{jh}) - S_{jh+1}(v_{jh} - \alpha, b_{jh}).
\]

That is,

\[
P_{jk}(b_{jk})[v_{jh} - \alpha - b_{jh}] \geq P_{jh+1}(b_{jh})[v_{jh} - \alpha - b_{jh}].
\]

So,

\[
P_{jh+1}(b_{jh}) \leq \frac{P_{jk}(b_{jk})[v_{jh} - \alpha - b_{jh}]}{v_{jh} - \alpha - b_{jh}}.
\]
Thus, replacing \( b_{jk'} - b_{jh} \geq \alpha \), and so this is at least

\[
\alpha \frac{m+3}{4\varepsilon^{m+1}} \quad \text{(by (9.3))}.
\]

Next, let \( T' \) be the number of bids for \( i \) between \( b_{ih} \) and \( b_{jl} \). Then, as before, the change in payoff to \( i \) from increasing \( b_{ih} \) to \( b_{jl} \) is

\[
S_{i,-r}(v_{il}, b_{jl}) - S_{i,-r}(v_{il}, b_{ih})
\]

\[
= S_{i,-r}(v_{il} - \alpha, b_{jl}) - S_{i,-r}(v_{il} - \alpha, b_{ih}) + \alpha (P_{jl}(b_{jl}) - P_{jl}(b_{jl} - \alpha))
\]

\[
\geq \alpha (b_{jl} - b_{jl} - \alpha) - \alpha (b_{ih} - b_{ih} - \alpha)
\]

\[
\geq \alpha \frac{m+3}{4\varepsilon^{m+1}} \quad \text{(by (9.3))}.
\]

To prove Lemma 6.1: Fix some large \( S \). Let \( n^* \) be such that \( \Pr(k^n < mS) < \varepsilon/2 \) for \( n > n^* \). Consider a realization in which \( k^n = k \geq mS \). If \( s^i < S \), then \( K_{s}^i(x) = 1 \) for all \( i, h \) and \( x \), and so the result is automatic. Assume \( s^i = s \geq S \). Let \( R(x) \) be the random variable giving the number of values for players other than \( i, j \) that lie at or above \( x \) (conditional on \( s^i = s \)). Let \( x^* = \min(x/R(x) \geq k) \).
Consider $x \geq x^* + \varepsilon/2$. By no asymptotic gaps, $E(R(x^*+\varepsilon/2)) < k - sM(\varepsilon/2)$. But $R(x^*+\varepsilon/2)$ is the sum of $s$ independent random variables having variance less than $m^3$, and so has variance less than $sm^2$. By Chebyshev's inequality,

$$\Pr(R(x^*+\varepsilon/2) > k - 2m) < \frac{sm^2}{(sM(\varepsilon/2) + 2m)^2} < \frac{sm^2}{(SM(\varepsilon/2) + 2m)^2}$$

(the second inequality is correct for large $S$) and is thus arbitrarily close to 0 for large enough $S$. That is, there are almost always enough objects to fill all values above $x^*$. That is, almost all the time, there are enough values at or above $x$.

So, $T_x$ is the surplus that a player would earn if he decentralized his choice of bids to $m$ agents bidding in separate auctions and facing the situation captured by $P^n(\cdot)$.
LEMMA 9.6: Assume \( A^n, e^n \) satisfies AES. Then, for all \( \varepsilon > 0 \), \( S_i^n(v) \to S_i^n(v) \) uniformly for all \( i \) and \( v \in V_r \).

PROOF: Note first that by AES, \( S_i^n(v, b) \to \sum_{n=1}^m T_i^n(v, b) \). This follows since \( |P_i^n(b) - P_i^n(b)| \to 0 \) and for the uniform case, since by the proof of Theorem 5.1 \( E(P_i) \) wins an \( h \)th object with \( \varepsilon \)th bid \( b \) does not depend much on either \( i \) or \( h \). But then, \( \max_{b \in B} S_i^n(v, b) \to \max_{b \in B} \sum_{n=1}^m T_i^n(v, b) \). Finally, note that for \( x' > x \) and \( b > b' \), \( T_i^n(x', b') \geq T_i^n(x', b) \) and so \( \max_{v \in [0, \pi]} T_i^n(v, b) \) is weakly increasing in \( x \). Hence, \( \sum_{n=1}^m \max_{b \in [0, \pi]} T_i^n(v, b) \) has a solution in \( B \),

\[
\max_{b \in B} \sum_{n=1}^m T_i^n(v, b) = \sum_{h=1}^m \max_{b \in [0, \pi]} T_i^n(v, b) = \sum_{h=1}^m T_i^n(v, b).
\]

That \( T_i^n(.) \) is convex is standard (as in the proof of Lemma 6.2).

LEMMA 9.7: Assume \( A^n, e^n \) satisfies AES. For all \( \varepsilon > 0 \), there is \( n^* = \infty \) such that for \( n > n^* \), for all \( \hat{v} \in V_r \),

\[
\frac{\partial T_i^n(x)}{\partial x} \bigg|_{x=\hat{v} + \varepsilon} - \varepsilon \leq \frac{\partial S_i^n(v)}{\partial v_h} \bigg|_{v=\hat{v}} \leq \frac{\partial T_i^n(x)}{\partial x} \bigg|_{x=\hat{v} + \varepsilon} + \varepsilon.
\]

\[21\]

The idea behind this lemma is that close-by convex functions have close-by slopes. Intuitively, since slopes are monotonically increasing, a violation of (9.8) implies a difference in slopes over a long enough interval to imply that the functions themselves are far apart.

PROOF: Let \( e_h \) be the \( m \)-vector with \( h \)th element 1, and other elements 0. Choose \( n^* \) such that \( |S_i^n(v) - S_i^n(v)| < \varepsilon^2/2 \) for all \( n > n^* \) and \( v \in V \). For some \( n > n^* \), and for \( \hat{v} \in V_r \), assume

\[
\frac{\partial S_i^n(v)}{\partial v_h} \bigg|_{v=\hat{v}} \geq \frac{\partial T_i^n(x)}{\partial x} \bigg|_{x=\hat{v} + \varepsilon} + \varepsilon.
\]

(The other case is analogous). Then, by convexity, and the definition of \( S_i^n \),

\[
\frac{\partial S_i^n(v)}{\partial v_h} \bigg|_{v=\hat{v} + \delta e_h} \geq \frac{\partial T_i^n(x)}{\partial x} \bigg|_{x=\hat{v} + \delta e_h} + \varepsilon = \frac{\partial S_i^n(v)}{\partial v_h} \bigg|_{v=\hat{v} + \delta e_h} + \varepsilon
\]

for all \( \delta \in [0, \varepsilon] \), where by the definition of \( V_r \), \( \hat{v} + \delta e_h \in V \) for \( \delta \in [0, \varepsilon] \). But then,

\[
\sum_{n=1}^m \max_{b \in [0, \pi]} T_i^n(v, b) \leq \sum_{n=1}^m T_i^n(v, b) + \varepsilon \geq \sum_{n=1}^m T_i^n(v, b) + \varepsilon^2/2,
\]

contradicting that \( |S_i^n(\hat{v}) - S_i^n(\hat{v})| < \varepsilon^2/2 \) and \( |S_i^n(\hat{v} + e_h) - S_i^n(\hat{v} + e_h)| < \varepsilon^2/2 \).

Q.E.D.

PROOF OF LEMMA 6.3: Assume one can find \( \varepsilon > 0 \) such that for \( n \) arbitrarily large there is \( i, \hat{v} \in V_r, \hat{b} \in B^n(\hat{v}) \), and \( h \) such that \( P_i^n(\hat{b}) \geq K_i^n(\hat{v} + \varepsilon) + \varepsilon \). (The case \( P_i^n(\hat{b}) \leq K_i^n(\hat{v} - \varepsilon) - \varepsilon \) is analogous.) We will show that this contradicts asymptotic efficiency.

Let \( \varepsilon^* \) be such that the number of value vectors expected to fall in \( V \setminus V_r \) is less than \( \varepsilon^* \). Such an \( \varepsilon^* \) exists by no asymptotic atoms (A4). Let \( \varepsilon' = \min(\varepsilon^*, \varepsilon/12) \). Let \( n^* \) be such that for all \( n > n^* \), \( \hat{v} \in V_r, \) and \( h \),

\[
\frac{\partial T_i^n(x)}{\partial x} \bigg|_{x=\hat{v} + \varepsilon'} - \varepsilon' \leq \frac{\partial S_i^n(v)}{\partial v_h} \bigg|_{v=\hat{v}} \leq \frac{\partial T_i^n(x)}{\partial x} \bigg|_{x=\hat{v} + \varepsilon'} + \varepsilon'.
\]

At kink points derivatives are interpreted as a selection from the set of slopes of tangents.
such that for all $x > e'$,

$$K_n^b(x) \geq K^n(x - e') - e'$$

and such that

$$E(a^n) / E(f^n) > 1 - \frac{3 e^2 M(e/4)}{2 \hat{v}}$$

where the existence of such an $n^*$ is guaranteed by Lemmas 9.7 and 6.1 and by Theorem 4.1. Let

$n > n^*, i, \hat{b}, \hat{b}$, and $h$ be defined as above.

Then, for all $v' \in V'_n$, $i'$, $h'$ such that $v''_{i'} \geq \hat{b} + 2 e^\star$, and $h' \in BR^h_i(v')$,

$$P_{h,i'}(h') = \frac{\partial S^n_b(v)}{\partial v_{i'}} \bigg|_{v = v'} \quad \text{(by Lemma (6.2))}$$

$$\geq \frac{\partial T^n(x)}{\partial x} \bigg|_{x = v' - e'} - e' \quad \text{(by 9.9)}$$

$$\geq \frac{\partial T^n(x)}{\partial x} \bigg|_{x = \hat{b} + e' - e'} - e' \quad \text{(by convexity of $T^n(\cdot)$)}$$

$$\geq \frac{\partial S^n_b(v)}{\partial v_{i'}} \bigg|_{v = \hat{b}} - 2 e' \quad \text{(by 9.9)}$$

$$= P_h(b') - 2 e' \quad \text{(by Lemma (6.2))}$$

$$\geq K^n(h + e) + e - 2 e' \quad \text{(by assumption)}$$

$$\geq K^n(h_i) (h + e - e') + e - 3 e' \quad \text{(by 9.10)}$$

$$\geq K^n(h_i) \left( \hat{b} + \frac{3}{4} e \right) + \frac{3}{4} e.$$

So, assume that for some $i'$ and $h'$, $v''_{i'} \in V'_n$, and $v''_{i'h'} \in (\hat{b} + (e/4), \hat{b} + (e/2))$. Then, in equilibrium, $i'$ wins an $h'$th object at least $(3/4) e$ of the time too much compared to efficiency. By no asymptotic gaps, $E(s^n) M(e/4)$ values are expected to fall in $(\hat{b} + (e/4), \hat{b} + (e/2))$, and hence $(1/2) E(s^n) M(e/4)$ in $(\hat{b} + (e/4), \hat{b} + (e/2)) \cap V'_n$. Hence, in expectation, $\frac{1}{2} E(s^n) M(e/4) (3/4) e$ too many objects are being allocated to such values. If $v''_{i'h'} \in (\hat{b} + (e/2), \hat{b} + (3e/4))$, then $i'$ again wins with probability greater than efficiency. It follows that in expectation, at least $\frac{1}{2} E(s^n) M(e/4) (3/4) e$ objects are misallocated from values above $\hat{b} + (3e/4)$ to values below $\hat{b} + (e/2)$, resulting in an expected efficiency loss of $(1/2) E(s^n) M(e/4) (3/4) e (e/4) = (3e^2/32) E(s^n) M(e/4)$. Since feasible surplus is bounded by $E(s^n) \epsilon$, it follows that $E(a^n) / E(f^n) < 1 - 3 e^2 M(e/4) / 32 \epsilon$ which contradicts (9.11).

Q.E.D.

**Proof of Theorem 6.4:** Fix an arbitrary $e > 0$. Using Lemma 6.3 and 9.7, choose $n^*$ such that for $n > n^*$, and $x \in (e, \tau - e)$,

$$K^n(x - e) - e < \frac{\partial T^n(x)}{\partial x} < K^n(x + e) + e.$$
Then, for any given \( \hat{x} \in [0,v] \),

\[
T^n(\hat{x}) = \int_0^\hat{x} \frac{\partial T^n(x)}{\partial x} \, dx
\]

\[
\leq \int_{x \in (0,\hat{x}) \cap (x, x+\varepsilon)} 1 \, dx + \int_{x \in (0,\hat{x}) \cap (x, x-\varepsilon)} \frac{\partial T^n(x)}{\partial x} \, dx
\]

\[
\leq 2\varepsilon + \int_{x \in (0,\hat{x}) \cap (x, x-\varepsilon)} \frac{\partial T^n(x)}{\partial x} \, dx
\]

\[
\leq 2\varepsilon + \int_{x \in (0,\hat{x}) \cap (x, x-\varepsilon)} K^n(x+\varepsilon) \, dx
\]

\[
\leq 2\varepsilon + \varepsilon \pi + \int_{x \in (0,\hat{x}) \cap (x, x-\varepsilon)} K^n(x) \, dx
\]

\[
\leq 3\varepsilon + \varepsilon \pi + \int_0^\hat{x} K^n(x) \, dx \quad \text{(since } 0 \leq K^n(x) \leq 1 \text{ always)}.
\]

Similarly,

\[
T^n(\hat{x}) \geq \int_0^\hat{x} K^n(x) \, dx - \varepsilon \pi - 3\varepsilon.
\]

The result follows from Lemma 9.6 since by integration by parts,

\[
\int_0^\hat{x} K^n(x) \, dx = K^n(\hat{x})(\hat{x} - E(y^n|y^n < \hat{x})). \quad \text{Q.E.D.}
\]

**Proof of Corollary 6.5:** Note that for \( b \) optimal for \( v' \),

\[
S^n_v(b) = \sum_{h=1}^m p^n_h(b_h)[v_h - b_h].
\]

Hence, by the limit characterization of revenue, it must be that for any \( \varepsilon' > 0 \), and for \( n \) large enough,

\[
\left| \sum_{h=1}^m p^n_h(b_h)[v_h - b_h] - \sum_{h=1}^m K^n(v_h)[v_h - E(y^n|y^n < v_h)] \right|
\]

\[
\approx \left| \sum_{h=1}^m K^n(v_h)[v_h - b_h] - \sum_{h=1}^m K^n(v_h)[v_h - E(y^n|y^n < v_h)] \right|
\]

\[
= \left| \sum_{h=1}^m K^n(v_h)b_h - E(y^n|y^n < v_h) \right| < \varepsilon'.
\]

For the single unit case, the result is thus obvious: if \( K^n(x) > \varepsilon \), then \( b_h - E(y^n|y^n < v_h) \) can be at most \( \varepsilon'/\varepsilon \). As \( \varepsilon' \) was arbitrary, we are done. For \( m > 1 \), one must rule out that for some \( h \) and \( h' \), \( b_h \ll E(y^n|y^n < v_h) \) for \( h \), while \( b_{h'} \gg E(y^n|y^n < v_{h'}) \) for \( h' \). The key to this is to observe that the surplus on unit \( h' \) must then be strictly below \( T(v_{h'}) \) (since \( b_{h'} \) is winning with the right probability, but paying too much). But, as in the proof of Lemma 4.4, \( i \) can then change \( b_{h'} \) to something optimal, without worrying much about the fact that he is changing the order of his bids, and in doing so earn close to \( T(v_{h'}) \), contradicting that \( b \) was optimal. \quad \text{Q.E.D.}
PROOF OF THEOREM 7.1 FOR THE DISCRIMINATORY CASE: For each \( x \) such that \( K^a(\cdot) \) is continuous at \( x \), define \( h^a(x) = E(y|^a < x) \). Consider \( x \) such that \( K^a(\cdot) \) jumps at \( x \).

Let \( Z = \Pr(y^a < x | x - E(y^a | y^a < x)) = \Pr(y^a < x | x - E(y^a | y^a < x)) \), and for each \( b \in (E(y^a | y^a < x), E(y^a | y^a < x)) \), let \( \hat{Z}(b) \) be defined such that \( \hat{Z}(b)(x - b) = Z \). \( \hat{Z}(b) \) is clearly strictly increasing in \( b, \hat{Z}(y^a < x) < \hat{Z}(y^a < x) \). But then, it is easily seen that there is a way for players with value \( x \) to symmetrically randomize over \( (E(y^a | y^a < x), E(y^a | y^a < x)) \) given that they have \( v_i = v \) such that \( P^a(b) = \hat{Z}(b) \) for each \( b \in (E(y^a | y^a < x), E(y^a | y^a < x)) \).

Let us establish that it is an \( \epsilon \)-equilibrium for all players to bid in this way. Consider any jump point \( x \). By construction, \( P^a(b)(x - b) \) is constant on \( E(y^a | y^a < x) < b \leq E(y^a | y^a < x) \), and equal to \( Z \). Let \( b > E(y^a | y^a < x) \) be a bid in the support of bids made by type \( w < x \). Then, the value of deviating to a bid of \( b' \) with value \( x \) is

\[
P^a(b')(x - b') - \Pr(y^a < w)(x - E(y^a | y^a < w)) \]

\[
+ \Pr(y^a < w)(x - E(y^a | y^a < w)) - \Pr(y^a < x)(x - E(y^a | y^a < x))
\]

By construction, the first term is 0 when \( x \) is replaced by \( w \), and so is weakly negative now. The second term simplifies to \( \Pr(x < y^a < w)(x - E(y^a | x < y^a < w)) \), which is again weakly negative. A similar argument applies to types \( w < x \).

So, consider \( v \in V \), and choose \( b^a \in B \) such that for each \( h, b^a_h \) is within the support of bids \( b^a \) specifies for \( v_h \). By construction, \( P^a(h)(v_h - b) \) is maximized at \( b^a_h \). But, by AES, it follows that \( b^a \) is within \( \epsilon \) of maximizing \( \max_{v \in V} \sum_i P^a_h(b)(v_h - b) \) for any \( i \), and so the specified behavior is an \( \epsilon \)-equilibrium.

Q.E.D.

REFERENCES


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