EFFICIENCY OF LARGE DOUBLE AUCTIONS

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We consider large double auctions with private values. Values need be neither symmetric nor independent. Multiple units may be owned or desired. Participation may be stochastic. We introduce a very mild notion of “a little independence.” We prove that all nontrivial equilibria of auctions that satisfy this notion are asymptotically efficient. For any $\alpha > 0$, inefficiency disappears at rate $1/n^{2-\alpha}$.

KEYWORDS: Double auction, efficiency, market microstructure, asymmetric information.

1. INTRODUCTION

Many market settings are approximated by a double auction. Standard examples are the London gold market and the order books maintained by New York Stock Exchange specialists. These auctions typically have many traders on each side of the market.

More importantly, large double auctions are an excellent model for microfoundations of price formation in competitive markets. Like a competitive market, a large double auction has many traders. However, unlike the standard competitive model, traders are strategic. Hence, if traders asymptotically ignore their effect on price, this is a result, not an assumption, and there is an explicit mechanism translating individual behaviors into prices. So one of the thorniest problems of the standard Walrasian model—how does the market get to equilibrium if everyone is a price taker—is explicitly addressed. Finally, double auctions are a better setting for thinking about price formation than one-sided auctions, both because they are often a better match to reality and, especially, because they capture the essential problems of trade better than a one-sided auction. A large one-sided auction allows one to ask if traded units end up in the right hands, but it does not address whether the correct number of units trade in the first place.

In a seminal paper, Rustichini, Satterthwaite, and Williams (1994, henceforth RSW) consider a double auction in which $n$ buyers and sellers draw private values independently and identically distributed. They show that symmetric, increasing, differentiable equilibria in this setting are in the limit efficient and that convergence is fast, of order $1/n^2$. This is especially attractive in light of experimental evidence on efficiency in double auctions with only a moderate number of players.\(^2\)

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\(^2\)Satterthwaite and Williams (2002) establish that in the i.i.d. setting, this rate is fastest among all mechanisms. Important precursors to RSW include Chatterjee and Samuelson (1983), Wilson (1985), Gresik and Satterthwaite (1989), and Satterthwaite and Williams (1989).
In independent work, Fudenberg, Mobius, and Seidel (2003) extend RSW to a setting in which a one-dimensional state is sampled and values are then drawn i.i.d. from a density that depends on the state, but has nonshifting support and uniform lower bound across states. They also show the existence of a pure increasing symmetric equilibrium when the number of players is large.

These results are useful in thinking about how auctions approximate competitive equilibria. However, there are several dimensions along which they could be strengthened.

1. The proof technique depends heavily on symmetric distributions of values.
2. Even in the symmetric setting, there is no guarantee of uniqueness. So, while well-behaved symmetric equilibria are asymptotically efficient, there may be other (possibly asymmetric) equilibria as well. In particular, there is always the no-trade equilibrium in which all buyers make an offer of zero and all sellers make an offer higher than any possible valuation. Results before this paper do not rule out other intermediate trade equilibria.
3. While one may be willing to rule out the asymmetric equilibria on a priori grounds in the symmetric case, selecting the “good” equilibria is much harder if the initial setting is itself asymmetric.
4. Imposing symmetry on values and bids assumes away half the problem. Objects that trade automatically move from and to the right people, and so the only question is whether the volume of trade is right. Without symmetry, it may also occur that, for example, a low-valued buyer wins an object when a higher-valued buyer does not.
5. Finally, these papers consider only single unit demands and supplies.

We present a model and results that address these points. We consider a generalized private value double auction setting. Players can be highly asymmetric and demand or supply multiple units. Beyond the assumption of private values, there are only three assumptions with any bite. First, while individual values need be neither full support nor even nondegenerate, we require that any given interval in the support of values is eventually hit in expectation by many players. We term this condition no asymptotic gaps (NAG). Analogously,
we require there to be \textit{no asymptotic atoms} (NAA): it cannot be the case that a positive limiting fraction of players are expected to pile up in an arbitrarily small interval.

Most critically, we drastically relax independence. We require only that a “little” independence across players persists as the number of players grows. A sequence of distributions over player values satisfies $z$-independence, $z \in (0, 1]$, if the probability of any given event on player $i$’s values changes by a factor bounded between $z$ and $1/z$ when one conditions on the values of the remaining players. The key requirement is that $z$ holds uniformly in the number of players. Two perfectly correlated random variables do not satisfy $z$-independence for any $z > 0$; 1-independence is the standard notion of independence.

An interpretation of $z$-independence is that each player has at least a small idiosyncratic component to his valuation, one that cannot be precisely predicted no matter how much one knows about the values of other players. This is a weak condition, admitting very broad classes of distributions. Because values can be highly correlated (positively, negatively, or otherwise) under $z$-independence, even in the limit the allocation and price setting problem will generally be nontrivial.

There is always a no-trade equilibrium in a double auction setting. Jackson and Swinkels (2005, henceforth JS) show that there is at least one nontrivial equilibrium as well, but because our setting is so general, these equilibria need not be in increasing or even pure strategies. Despite this, our major result is simply stated:

\textbf{RESULT:} As the number of players grows, every nontrivial equilibrium of the double auction setting converges to the Walrasian outcome. Inefficiency disappears at rate $1/n^{2-\alpha}$ for any $\alpha > 0$.

Asymptotic efficiency implies asymptotic uniqueness and pureness: over relevant ranges, bids must be arbitrarily close to value. Thus, as $n$ grows large, there are precisely two types of equilibria of private value double auctions:

1. Equilibria that involve no trade.
2. Equilibria in which a near efficient level of trade occurs, at a price near the competitive one.

With single-unit demands and supplies, our proof works because in each outcome of a double auction, there is at most one buyer who is both currently winning an object and who would have raised the price had he bid more (the lowest winning buyer). So, while many buyers might have raised price by bidding more, only one would care that he did so. This is symmetric for sellers considering lowering their bids. So, the expected relevant impact on price from increased bids by buyers is already order $1/n$. In addition, even if an increase in a bid increases price, it should do so by an amount related to $1/n$, since this should be the expected distance to the next bid. But then, since the expected
impact on price is order $1/n^2$, it must be that bidding honestly almost never
wins an extra object and so those objects that are traded must be allocated
very efficiently.

The focus then turns to showing that the right number of objects trade, or,
equivalently, that the competitive gap defining the range of market clearing
prices grows small. This turns out to be much the hardest part of the paper
(especially with a rate). In the symmetric case, one can appeal to the first-
order conditions of players near a discontinuity in bids. Here, things are much
more difficult, because without symmetric increasing strategies, (a) the very
concept of a “gap” becomes more complicated, (b) it is hard to identify which
player types might bid near a gap, and (c) players can have very different be-
liefs about the likelihoods of the events involved. We show that the only way
to have a significant competitive gap without violating the efficiency already
shown for those objects traded is for the market to essentially become deter-
ministic, with a given set of buyers and sellers always trading. Any member of
either of these groups can then favorably influence the price without losing the
chance to trade.

The efficiency result generalizes to multiple-unit demands as long as NAG
continues to hold for the first unit of demand and supply for each player. If
this holds, we can reformulate the arguments just outlined but applied only to
the highest bid by each buyer and lowest bid by each seller to show small price
impacts of honest bidding. From there to (fast) efficiency for all units involves
a careful tracking of incentives, but is otherwise straightforward.

We begin by setting up the basic single-unit demand and supply model. We then introduce $z$-independence. Analysis of efficiency for the large dou-
ble auction with single-unit demands and supplies follows. Then we generalize
to auctions with multiple-unit demands and supplies. We conclude with some
thoughts on extensions. All proofs are relegated to the Appendix.

2. THE MODEL

We begin with the structure of a given double auction $A$. A finite set $N$ of
players is divided into subsets $N_S$ and $N_B$. Players in $N_S$ are potential sellers,
each with one unit to sell. Players in $N_B$ are potential buyers, each desiring a
single unit.

Each $i \in N$ has valuation $v_i$. For sellers, this might be either a production
cost or a value in use. For $i \in N_B$, we assume $v_i \in [0, 1)$. For $i \in N_S$, we assume
$v_i \in (0, 1]$. Ruling out buyers with value 1 and sellers with value 0 implies that
a buyer with value 0 or a seller with value 1 will never trade. Hence, one can
“park” extra buyers at 0 and extra sellers at 1. There is no loss of generality in
assuming an equal number of buyers and sellers. Let $n \equiv |N_S| = |N_B|$. Because
extra buyers and sellers can be parked, the model also allows a stochastic (but
bounded) number of buyers and sellers. The vector $v \equiv \{v_i\}_{i \in N}$ is drawn accord-
ing to a Borel probability measure $P^n$ on $[0, 1)^n \times (0, 1]^n$. The marginal of $P^n$
on $v_i$ is $P^n_i$ and the marginal onto $v_{-i}$ is $P^n_{-i}$.
Throughout the paper, for any nonempty \( K \subset N \), when we write \( F_K \) (respectively, \( F_i \), \( F_{i-1} \), \( F_{N\setminus K} \)), we mean an arbitrary positive probability Borel event that involves only the values or bids of the players in \( K \) (respectively, \( \{i\} \), \( N \setminus i \), \( N \setminus K \)). For events \( F_i \subset [0, 1] \) and \( F_{i-1} \subset [0, 1]^{2^n-1} \) we will let \( P_n(F_i|F_{i-1}) \) be the conditional on \( i \)'s values.\(^6\)

Each player \( i \) observes his value and then submits a bid \( b_i \in [0, 1] \). Trade is determined by crossing the submitted demand and supply curves. Call the (random) range of possible market clearing prices the competitive gap, \( c_g \equiv [c_g, \overline{c_g}] \). If we let \( b_j \) denote the \( j \)th highest bid, then a little time with the appropriate figure (Figure 1) shows that \( c_g = [b^{n+1}, b^n] \).

**ASSUMPTION 1:** Trade takes place at price

\[
p = \hat{p}(c_g, \overline{c_g}),
\]

where \( \hat{p} \) is differentiable, takes values in \( [c_g, \overline{c_g}] \), and has derivatives bounded by 0 and 1.\(^7\)

Imagine that the bidder who submitted \( \overline{c_g} \) raises his bid. As long as his bid continues to define \( \overline{c_g} \), Assumption 1 ensures that he raises the price at rate at most 1. As soon as he passes the next bid up, he ceases to affect the price. Let

![Figure 1](image)

**FIGURE 1.**—The submitted demand and supply curves. Note there are a total of \( n \) buy and sell bids at or above \( \overline{c_g} \).

\(^{6}\)In principle all of these objects depend on \( n \). We suppress the superscript wherever possible.

\(^{7}\)This, of course, includes the standard \( k \) double auction.
\( \bar{ug} \equiv b^{n-1} \) be this next bid and define the upper supporting gap as \( ug \equiv [\bar{g}, \bar{ug}] \). Then, the maximum effect on the price is \(|ug|\). Similarly, let \( lg \equiv b^{n+2} \) and define the lower supporting gap as \( lg \equiv [lg, cg] \). A bidder who lowers his bid ceases to affect the price as soon as \( lg \) is passed. So, \( cg \) determines the amount of choice there is in setting a market price, while \( lg \) and \( ug \) determine how closely “supported” this range is.

Each player \( i \) has a von Neumann–Morgenstern (vNM) utility function \( u_i \) with slope bounded away from 0 and \( \infty \). Thus no particular structure on risk preferences is required.\(^8\)

2.1. Equilibrium

A set of distributional strategies \( \{\mu_i\}_{i \in N} \) (Milgrom and Weber (1985)) is an equilibrium if it is a Bayesian Nash equilibrium in which buyers never bid above \( v_i \), and sellers never bid below \( v_i \). So, our definition of equilibrium encompasses a weak dominance requirement. The equilibrium is nontrivial if there is a positive probability of trade.

We show that nontrivial equilibria are asymptotically efficient. This, of course, is a better result if such equilibria exist! Under slightly stronger conditions than we use here, JS show that this is indeed the case.\(^9\)

2.2. Sequences of Auctions

Consider a sequence of such auctions \( \{A^n\} \), where \( n \) tends to infinity. We need three conditions that apply across \( n \). First, while individual values need not have full support (and may, in fact, be atomic), we require that as \( n \) grows large, each subinterval is hit with nonvanishing probability.

**ASSUMPTION 2 —No Asymptotic Gaps:** There is \( w > 0 \) such that for all \( n \) and for all intervals \( I \subseteq (0, 1) \) longer than \( 1/n \),

\[
\sum_{i \in NB} P^n_i[I] \geq wn|I|
\]

and

\[
\sum_{i \in NS} P^n_i[I] \geq wn|I|.
\]

\(^8\)In the proofs, we assume risk neutrality. Dealing with vNM utility functions with slope bounded away from 0 and \( \infty \) involves scaling potential gains and losses by some factor from the risk neutral case. This merely introduces notation.

\(^9\)The two key assumptions are that \( P \) have a Radon–Nikodym derivative with respect to \( \prod_i P_i \) that is bounded away from 0 and \( \infty \), and that each \( P_i \) is atomless. Neither assumption plays any further role in the development here.
Our second assumption similarly requires that not too many values fall in a given interval.

**Assumption 3 —No Asymptotic Atoms:** There is \( W < \infty \) such that for all \( n \) and for all intervals \( I \subseteq (0, 1) \) longer than \( 1/n \),

\[
\sum_{i \in \mathcal{N}_B} P^n_i[I] \leq Wn|I|
\]

and

\[
\sum_{i \in \mathcal{N}_S} P^n_i[I] \leq Wn|I|.
\]

These conditions hold only on \((0, 1)\), allowing a positive mass of buyers with value 0 or sellers with value 1, consistent with our earlier discussion of “parking” extra players.

**Example 1:** Let sellers \( i \in \{1, \ldots, n\} \) have \( v_i = i/n \) and similarly for buyers. Conditions NAG and NAA are satisfied for \( w = W = 1 \). So, individual values need neither have full support nor be nonatomic.

Each of these two assumptions has an analogue in RSW. The conditions NAG and NAA are needed for a rate of convergence result, but not for convergence itself (see Section 6).

### 3. \( z \)-Independence

Our final condition is the most important. We wish to relax independence considerably while still requiring “some persistent independence” as the population grows.

We require that knowledge about the values of players other than \( i \) provides at most a finite likelihood ratio on the value of player \( i \), independent of how many other players there are.

**Definition 1:** The sequence of probability measures \( \{P^n\} \) satisfies \( z \)-independence, \( z \in (0, 1) \), if for every \( n \), for all \( i \in N \), for any positive probability events \( F_{-i}, F'_{-i} \) involving only \( v_{-i} \) and any positive probability event \( F_i \) involving only \( v_i \),

\[
z \leq \frac{P^n_i(F_i|F_{-i})}{P^n_i(F'_i|F'_{-i})} \leq \frac{1}{z}.
\]
That is, there is still some idiosyncrasy in each $v_i$ even as the market becomes large.\footnote{A contemporaneous paper by Peters and Severinov (2006) uses a similar condition (in a different model) in a finite type setting. This condition is related to the notion of $\psi$-mixing described in Bradley (2005).}

The real content of $z$-independence is in the uniformity of $z$ across $n$. For fixed $n$, $z$-independence is stronger than mutual absolute continuity of $P^n$ with respect to the product measure $\prod_{i \in N} P^n_i$ (consider a setting where, depending on $v_1$, $v_2$ is distributed according to either a uniform or a triangular distribution on $[0, 1]$), but weaker than having a continuous Radon–Nikodym derivative bounded from 0 and $\infty$.\footnote{An equivalent definition of $z$-independence is that for every $i$ and $n$, the Radon–Nikodym derivative of $P^n$ with respect to $P^n_i \times P^n_{-i}$ exists and is essentially bounded above by $1/z$ and below by $z$.}

**Assumption 4** — $z$-Independence: There exists $z \in (0, 1]$ such that \{\$P^n\$\} is $z$-independent.

### 3.1. Examples

We begin with an example that fails $z$-independence.

**Example 2:** With probability $1/2$, values are drawn i.i.d. uniform $[0, 1]$ and with probability $1/2$, $x$ is drawn uniformly from $[1/n, 1 - 1/n]$, and values are drawn i.i.d. uniform $[x - 1/n, x + 1/n]$. For each $n$, $P^n$ is absolutely continuous with respect to $\prod P_i$ (and, the example is easily modified such that the Radon–Nikodym derivative is continuous as well). However, as $n \to \infty$, seeing the values of any two given players within $2/n$ of each other makes it arbitrarily likely that any given remaining player will also have such a value and so $z$-independence is not satisfied.

Next are several examples that do satisfy $z$-independence. The first illustrates the importance of applying $z$-independence one player at a time. Even under $z$-independence, a summary statistic about a large group of players can be arbitrarily informative about a summary statistic about the rest of the players.

**Example 3:** Nature chooses $x \in \{L, H\}$ equiprobably. If $L$ is drawn, values are drawn i.i.d. according to density $f(v) = 1/2 + v$. If $H$ is drawn, values are drawn i.i.d. according to density $f(v) = 3/2 - v$.

Let $F_O$ be the event that less than 50\% of the odd numbered buyers have value below 1/2 and let $F_E$ be the event that less than 50\% of the even numbered buyers have value below 1/2. Then, as $n \to \infty$, $\Pr(F_O \cap F_E^c) \to 0$. However, under the product of the marginals, $\prod_{i \in N} P^n_i (F_O \cap F_E^c) = \frac{1}{4}$. In particular
then, the Radon–Nikodym derivative of $P_n$ with respect to $\prod_{i \in N} P^n_i$ grows arbitrarily large as $n$ increases.

However, $1/3$-independence is satisfied for this example: all one can extract from $v_{-i}$ is information about whether $x$ is $L$ or $H$, which changes the density on $v_i$ from 1 to something between 1/2 and 3/2.

Example 3 generalizes to any process in which a state is sampled and then, conditional on the state, values $v_i$ are drawn independently from measures with nonmoving support $V_i$ according to densities uniformly bounded (across states and $n$) away from zero and infinity. So our setting encompasses Fudenberg, Mobius, and Seidel (2003) (and more importantly, nonsymmetric analogues to their model).

Postlewaite and Schmeidler (1986) define nonexclusivity as a situation where the information of $n - 1$ players is enough to predict the relevant state of the economy. A variety of follow-on papers relax this to hold only asymptotically. On first view, $z$-independence is antithetical to nonexclusivity, since no matter how much is known about the rest of the players, the value of player $i$ remains uncertain. However, note that nonexclusivity refers to information about the underlying state, not to the signals players realize conditional on those states. In Example 3, $v_{-i}$ is asymptotically fully informative about $L$ versus $H$, while of bounded informativeness about $v_i$. Hence Example 3 satisfies both conditions.

**EXAMPLE 4:** Nature draws $v_1$ uniformly from $[0, 1]$ (this person is a “fashion leader”) and then draws subsequent players i.i.d. according to a density with support $[0, 1]$ but concentrated around $v_1$.

Since the impact of an early draw on later draws does not vanish, $z$-independence does not imply weak mixing. It is also easy to construct sequences that satisfy weak mixing under which successive draws are arbitrarily correlated, violating $z$-independence.

**EXAMPLE 5:** A parameter $x$ is chosen from $[0, 1]$. Values are drawn conditionally independently according to $f(\cdot|x)$, where $f(\cdot|x)$ satisfies the monotone likelihood ratio property (MLRP) in $x$. As long as $f(\cdot|0)/f(\cdot|1)$ is uniformly bounded, $z$-independence is satisfied for $z = \min_x f(x|0)/f(x|1)$. Choose a subset of the players and replace $v_i$ by $1 - v_i$. This measure continues to satisfy $z$-independence, but is obviously not affiliated. So, affiliation has essentially nothing to do with the issues at hand.

We close this subsection with an example that illustrates the surprising degree of correlation $z$-independence permits. Define $[x]$ as the largest integer smaller than $x$.

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12A good entry point is McLean and Postlewaite (2002).
EXAMPLE 6: For \( m \leq n \), let \( \zeta_B(m) = \binom{m}{n} (0.5)^n \) be the probability of \( m \) heads from flipping \( n \) fair coins.\(^{13}\) Now, for some \( 0 < a < 1/2 \), generate \( \zeta_C \) from \( \zeta_B \) by first defining \( \zeta'_C(m) = \zeta_B(m) a^{m - \lfloor n/2 \rfloor} \) and then defining \( \zeta_C \) from \( \zeta'_C \) by normalizing. Informally one makes each outcome successively further away from \( \lfloor n/2 \rfloor \) more unlikely by a factor of \( a \). Choose \( m \) according to \( \zeta_C \), choose each subset of coins of size \( m \) with equal probability, and make the \( m \) coins in the subset heads and the remainder tails. When \( a \) is small, drawing exactly \( \lfloor n/2 \rfloor \) heads by this process becomes very probable.\(^{14}\) For \( a = .1 \), e.g., there is an 80\% chance of exactly \( \lfloor n/2 \rfloor \) heads regardless of \( n \).\(^{15}\)

Nonetheless this process satisfies \( a^2 \)-independence. If there are \( m' \) heads among all but coin \( i \), the probability that \( i \) is heads is \( \Pr(m = m' + 1) / \Pr(m = m' + 1) \). By construction, this is either \( a \) or \( 1/a \).

Under \( z \)-independence, probabilities that start in the interior of \((0, 1)\) cannot be moved too far toward or away from the boundaries. However, as this example illustrates, probabilities can be moved around essentially arbitrarily otherwise.

The techniques in Mailath and Postlewaite (1990), Al-Najjar and Smorodinsky (1997), and Swinkels (2001) all rely on there being “noise” in the sense that the exact number of players who take a specific action (say bidding above some threshold) becomes diffuse. Example 6 illustrates that those techniques do not apply here.

More strikingly, in Example 1, there is no uncertainty at all. However, 1-independence is clearly satisfied. More generally, our results apply to any deterministic environment that satisfies NAA and NAG.

It is also worth noting that the deterministic environment is a clean example where strategies are mixed, but asymptotic efficiency obtains nonetheless.

\(^{13}\)The example can easily be extended from coins to values in the standard domain.

\(^{14}\)Note that

\[
\sum_{m=0}^{r} \zeta'_C = \sum_{m=0}^{r} \zeta_B(m) a^{m - \lfloor n/2 \rfloor} \leq \zeta_B \left( \frac{r}{2} \right) \sum_{m=0}^{r} a^{m - \lfloor n/2 \rfloor} \\
\leq \zeta_B \left( \frac{r}{2} \right) \left( 1 + 2 \sum_{i=1}^{\infty} a^i \right) = \zeta_B \left( \frac{r}{2} \right) \left( 1 + \frac{2a}{1-a} \right).
\]

Thus

\[
\zeta_C \left( \frac{n}{2} \right) \geq \frac{\zeta_B \left( \frac{n}{2} \right)}{\zeta_B \left( \frac{n}{2} \right) \left( 1 + \frac{2a}{1-a} \right)} = \frac{1}{\left( 1 + \frac{2a}{1-a} \right)}.
\]

As \( a \to 0 \), this tends to 1.

\(^{15}\)For \( a = 0.1 \), the previous expression is equal to \( 1 / (1 + (2(0.1))/(1 - 0.1)) \approx 0.81 \).
3.2. A Preliminary Lemma

Our first lemma shows that if values are \( z \)-independent, then so too are bids. The intuition for this is that observing a player’s bid is at most as informative as observing his type.\(^{16}\) It also describes the implications of \( z \)-independence for groups of players.

**Lemma 1:** Fix a nonempty \( K \subset N \). Let \( a = \min\{|K|, |N \setminus K|\} \). Then for all \( F_K \) and \( F_{N \setminus K} \),

\[
(2) \quad z^{-a} \Pr(F_K) \geq \Pr(F_K|F_{N \setminus K}) \geq z^a \Pr(F_K).
\]

Let \( X_K \) be a random variable that depends only on the values/bids of the players in \( K \). Then

\[
(3) \quad z^{-a} E(X_K) \geq E(X_K|F_{N \setminus K}) \geq z^a E(X_K).
\]

When \( a \) is large, these bounds are weak; for events that involve many players, likelihood ratios can explode.

3.3. Large Deviations

Given \( K \subset N \) and events \( \{F_i\}_{i \in K} \), let \( Q_K \) be the number of \( F_i \) that occur. We would like to understand the properties of \( Q_K \). In this section, we first relate the stochastic behavior of \( Q_K \) to a set of independent coins. Lemma 2 exploits the well understood large deviation properties of sets of independent coins to derive stochastic bounds on \( Q_K \).

Note first that for each \( i \),

\[
\Pr(F_i|F_{-i}) \geq z \Pr(F_i).
\]

Also

\[
(4) \quad \Pr(F_i^c|F_{-i}) \leq \frac{1}{z} \Pr(F_i^c) = \frac{1}{z} (1 - \Pr(F_i))
\]

and so

\[
\Pr(F_i|F_{-i}) \geq 1 - \frac{1}{z} (1 - \Pr(F_i)).
\]

Thus,

\[
\Pr(F_i|F_{-i}) \geq p_i \equiv \max \left\{ z \Pr(F_i), 1 - \frac{1}{z} (1 - \Pr(F_i)) \right\}.
\]

\(^{16}\)A related lemma appears in JS.
Since this is true for all $F_{-i}$, we will show that for any given $F_{N\setminus K}$, $Q_K$ first-order stochastically dominates (FOSD) $|K|$ independent coins with parameters $p_i$.

Similarly, for any given $F_{N\setminus K}$, $|K|$ independent coins with parameters $p_i$ stochastically dominate $Q_K$.

We then have the following application of large deviations:

**Lemma 2:** For all $K \subset N$ and $F_{N\setminus K}$,

$$\Pr(Q_K < \frac{z}{3} E(Q_K) \mid F_{N\setminus K}) \leq e^{-0.3z E(Q_K)},$$

$$\Pr(Q_K > \frac{3}{z} E(Q_K) \mid F_{N\setminus K}) \leq e^{-E(Q_K) E(Q_K \mid F_{N\setminus K})}.$$

The logic underlying Lemma 2 also implies that the probability of at least one success in $K$ is not drastically affected by $F_{N\setminus K}$.

**Corollary 1:** For all $K \subset N$ and $F_{N\setminus K}$

$$\Pr(Q_K \geq 1 \mid F_{N\setminus K}) \geq (1 - e^{-z}) \Pr(Q_K \geq 1).$$

### 3.4. Normal Realizations

We prove convergence at rate $1/n^{2-\alpha}$ for any given $\alpha > 0$. It is convenient to fix $\alpha$ now. We will need various fudge factors along the way. Choose $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that

$$\alpha > \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 > 2\alpha/3.$$

Let $w' = \frac{6}{\alpha} w$ and $W' = \frac{6}{\alpha} W$.

**Definition 2:** A realization of $v = \{v_i\}_{i \in N}$ is normal if every interval $I \subseteq (0, 1)$ longer than $1/n^{1-\alpha/3}$ has between $w'n|I|$ and $W'n|I|$ buyers with value in that interval and between $w'n|I|$ and $W'n|I|$ sellers with value in that interval.

Let $\mathcal{N}$ be the event that the realization is normal. We use SL as an abbreviation for “sufficiently large.” A key implication of Lemma 2 is the following lemma:

**Lemma 3:** For all $n$ SL, $\Pr(\mathcal{N}) \geq 1 - 1/n^4$. 

Together with NAG and NAA, Lemma 3 implies that the limiting realized true demand and supply curves are unlikely to have either vertical or flat sections (except at 0 for buyers and 1 for sellers).

3.5. Comments on the Rate of Convergence

Our major result is to show that percentage efficiency losses are asymptotically less than $1/n^{2-\alpha}$. As for all rate of convergence results, this does not say anything about small $n$. A weakness of our results is that the construction that underlies normality only holds for $n$ extremely large.

At many other points in the paper, we sacrifice the size of the constant for transparency of exposition: to a certain extent, this same practice is reflected here. It is also the case that normality is very demanding. In particular, we require that even the emptiest interval has many values in it, despite the fact that these intervals are shrinking nearly at rate $1/n$.

It is, in fact, irrelevant whether most intervals have values in them or not. What matters is just the intervals near the competitive and supporting gaps. Without further structure on the equilibrium or statistical setting, we do not see any way to rule out that the universe conspires against us in such a way that the emptiest interval always happens to be the relevant one. So, we use the rather crude technique of making even the emptiest interval full. It is an open question what else one would need to sidestep this technique.

There seems to be nothing in the underlying incentives being exploited that precludes a small number being the $n$ that is “sufficiently large” for the $1/n^{2-\alpha}$ rate to become effective. Rustichini, Satterthwaite, and Williams supplement their rate result (where the constant is also large) with numerically solved examples. Such solutions are beyond our ability in this setting.

If one is satisfied to simply show convergence without a rate, then none of the infrastructure of normality is relevant.

4. ANALYSIS OF THE DOUBLE AUCTION

4.1. Summing Deviations

Fix an equilibrium $\mu$ of $A^n$. Consider buyer $i$’s distributional strategy $\mu_i$. A deviation for $i$ is a measurable mapping $d_i$ from $[0, 1]^2$ to $[0, 1]$. First $i$ draws $v_i$ and $b_i$ according to $\mu_i$, but then he modifies his chosen bid according to $d_i$. Consider $d_i$ for which $b_i \leq d_i(b_i, v_i) \leq v_i \forall b_i, v_i$. That is, $i$ sometimes raises his bid, but not beyond his true value (since $\mu_i$ did not involve $i$ bidding more than her true value, this is coherent).

In any given realization, $d_i$ may have benefit $\hat{B}_i$, in that $i$ wins when he otherwise would not have, or may have cost $\hat{C}_i$, in that $i$ pays more when he would have already won. To formalize this, let $p$ be the price under $\mu$, and let $p_d$ be
the price when \( i \) uses \( d_i \). Let \( W_i \) be the event that \( i \) wins with \( d_i \), but not without. So \( \hat{B}_i = v_i - p_d \) when \( W_i \) occurs and is 0 otherwise. Let

\[
B_i \equiv E(\hat{B}_i) = \Pr(W_i)E(v_i - p_d | W_i)
\]

be the expectation of \( \hat{B}_i \).

Similarly, let \( O_i \subset W_i^c \) be the event that \( i \) wins without \( d_i \). Then \( \hat{C}_i = p_d - p \) when \( O_i \) occurs and is 0 otherwise. Let

\[
C_i \equiv E(\hat{C}_i) = \Pr(O_i)E(p_d - p | O_i).
\]

Since \( \mu_i \) is a best response, \( B_i \leq C_i \). So, given such a deviation \( d_i \) for each buyer,

\[
\sum_{NB} B_i \leq \sum_{NB} C_i.
\]

Each \( d_i \) is unilateral, but there is nothing wrong with summing the incentive constraints implied.

Let \( T \) be the event that trade occurs. Consider \( \sum_{NB} C_i \). Ex post, \( \hat{C}_i > 0 \) only if (a) trade was occurring (event \( T \)) and (b) the original \( b_i \) was equal to \( cg \), and uniquely so. To see this, note that when \( b_i > cg \) (or is tied at \( cg \)), increasing \( b_i \) does not affect \( p \) (which is a function only of \( cg \) and \( cg \)). If \( b_i < cg \), increasing \( b_i \) may increase \( p \), but since \( i \) was not originally winning, he is unhurt. So there is at most one \( i \) with \( \hat{C}_i > 0 \),\(^{17}\) and, as discussed above, for this \( i \), \( \hat{C}_i \leq |ug| \). Thus,

\[
\sum_{NB} C_i \leq \Pr(T)E(|ug| | T).
\]

For sellers, the same analysis applies if bids are lowered, but not below value. We have thus established:

**Lemma 4:** For any set \( \{d_i\}_{i \in NB} \) of deviations for which \( d_i(b_i, v_i) \in [b_i, v_i] \) for all \( (b_i, v_i) \),

\[
\sum_{NB} B_i \leq \sum_{NB} C_i \leq \Pr(T)E(|ug| | T).
\]

For any set \( \{d_i\}_{i \in NS} \) for which \( d_i(b_i, v_i) \in [v_i, b_i] \) for all \( (b_i, v_i) \),

\[
\sum_{NS} B_i \leq \sum_{NS} C_i \leq \Pr(T)E(|lg| | T).
\]

\(^{17}\) If \( cg \) is a seller’s bid, no buyer is hurt by \( d_i \).
While easy to prove, this bound is powerful. Independent of the number of bidders, the total benefit of making new trades by bidding more aggressively must be small in equilibrium. Note also that Lemma 4 remains true if one replaces $T$ by any $T' \supseteq T$.

4.2. The Probability of Trade Is Bounded from Zero

An important first step is to show that nontrivial equilibria are not “almost trivial” in the sense that trade becomes increasingly rare as $n$ grows. For each $n$, choose a nontrivial equilibrium of $A^n$. Let $V$ be the number of objects traded and recall that $T = \{V \neq 0\}$. As before, $V$ is a random variable, the distribution of which depends on both $n$ and the equilibrium under consideration. We suppress this in the notation.

**Proposition 1:** There is $\gamma > 0$ such that for all $n SL$ and all nontrivial equilibria, $\Pr(T) \geq \gamma$.

For intuition, say that a buyer’s offer is serious if it is above (say) $\frac{1}{2}$ and a seller’s if it is below $\frac{1}{2}$. Assume that the probability of even one serious buy offer is some $\delta$, which is positive but close to zero, and similarly for sellers (clearly, if there is a nonvanishing probability of a serious offer on either side, trade will not disappear). Trade occurs at most $2\delta$ of the time, since trade requires a serious offer from at least one side. Hence, by Lemma 4, the total costs to buyers (or sellers) of making more generous offers is like (has the same order as) $\delta$, but from $z$-independence and Corollary 1, the probability of a serious offer from one side but not the other is like $(1 - \delta)\delta \approx \delta$. When there is a serious offer on one side but not the other, a number of bidders on the other side that grows like $n$ would have benefited by deviating to trade at the serious offer. The aggregate gains are thus like $n\delta$, while costs are like $\delta$. This is a contradiction.

4.3. Small Supporting Gaps

We show next that the upper and lower supporting gaps shrink quickly. This will mean that the bounds in Lemma 4 are very powerful.

First, we show that $E(|ug|)$ (respectively, $E(|lg|)$) is like $1/n$. The idea is most easily seen if for each $n$, $ug$ has constant length $x$. By Lemma 3, a number of buyers proportional to $nx$ will have $v_i$ in the top half of $ug$. At most one of these buyers is winning an object (they are not bidding above $\overline{ug}$, because bids are below value, and at most one bid below $\overline{ug}$ is filled). By raising $b_i$ to $v - x/2$, all but this player (acting unilaterally) would win an extra object and earn at least $x/2$.\textsuperscript{18} So $\sum B_i \geq nx^2$ (up to some constants), but by the argument

\textsuperscript{18}Note that this will be true even if under the original realization of bids there was no trade at all.
preceding Lemma 4, \( \sum C_i \leq x \), since the one person who is hurt raises the price by at most \(|ug|\). Thus

\[
    nx^2 \leq \sum_{N_B} B_i \leq \sum_{N_B} C_i \leq x,
\]

from which \( x \leq \frac{1}{n} \). The actual proof has to account for the fact that \(|ug|\) is stochastic, as are the number of bidders in any given interval. Formally:

**Lemma 5:** For \( n \) SL,

\[
    E(|ug|) \leq \frac{1}{n^{1-\alpha_4}}, \quad E(|lg|) \leq \frac{1}{n^{1-\alpha_4}}.
\]

Having bounded the expectation of \(|ug|\), we can get a better handle on its distribution. Fix \( x \) and consider \( \Pr(|ug| \geq x) \). Consider the deviation in which buyers raise \( b_i \) to \( v_i - x/2 \). When \(|ug| \geq x\), then as above, a number of buyers like \( nx \) make gains \( x/2 \) and so \( \sum B_i = nx^2 \) (again ignoring constants). In addition, \( \sum C_i \leq E(|ug|) \leq 1/n \) from the first step. So

\[
    \Pr(|ug| \geq x)nx^2 \leq \frac{1}{n},
\]

from which \( \Pr(|ug| \geq x) \leq 1/(n^2x^2) \). Formally:

**Lemma 6:** For \( n \) SL and all \( x \),

\[
    \Pr(|ug| \geq x) \leq \frac{1}{n^{2-\alpha_3}x^2}, \quad \Pr(|lg| \geq x) \leq \frac{1}{n^{2-\alpha_3}x^2}.
\]

The preceding two lemmas allow us to derive a tight bound on inefficiencies caused by misallocating the objects that actually trade (whether the right number of objects trade is the subject of the next section). For \( x \geq 0 \), let \( L_B(x) \) be those buyers with values above \( \overline{cg} + x \) who do not receive an object and let \( l_B(x) \equiv \#L_B(x) \). Similarly let \( L_S(x) \) be those sellers with values below \( cg - x \) who do not sell and let \( l_S(x) \equiv \#L_S(x) \). Let

\[
    Y_B(x) \equiv \sum_{i \in L_B(x)} v_i - \overline{cg}, \quad Y_S(x) \equiv \sum_{i \in L_S(x)} cg - v_i.
\]

For buyers with values above \( \overline{cg} + x \), this is the loss in consumer surplus compared with being able to price take at \( \overline{cg} \), and analogously for sellers. Our next lemma uses Lemma 5 to show that both the number of such players and the associated loss is small. The intuition again comes from considering players bidding closer to their values.
**LEMMA 7:** For $n$ SL and for all $x$,
\[
E(l_B(x)) \leq \frac{1}{x n^{1-a_4}}, \quad E(l_S(x)) \leq \frac{1}{x n^{1-a_4}}.
\]

Furthermore,
\[
E\left(Y_B\left(\frac{1}{n}\right)\right) \leq \frac{1}{n^{1-a_3}}, \quad E\left(Y_S\left(\frac{1}{n}\right)\right) \leq \frac{1}{n^{1-a_3}}.
\]

### 4.4. Small Competitive Gaps

Let us now turn to the competitive gap. Our key lemma turns out to be much the hardest to prove:

**LEMMA 8:** For $n$ SL and for all $x$,
\[
\Pr(|cg| \geq x) \leq \frac{1}{n^{2-a_2} x^2}.
\]

To see the idea behind Lemma 8, consider the simpler situation in which there is some interval $I = (\bar{I}, \tilde{I})$ of length $x$ such that nobody ever bids in $I$ and such that $\Pr(I \subseteq cg)$ does not vanish quickly.

Imagine that we can show that this implies that $\Pr(I \subseteq cg) \rightarrow 1$. Then, by bidding $I + \epsilon$, any buyer who used to bid above $\bar{I}$ can still trade almost as often and force the price near $I$, while by bidding $\tilde{I} - \epsilon$, any seller who used to bid below $\bar{I}$ can still trade almost as often and force the price near $\tilde{I}$. One or the other of these must be profitable, contradicting equilibrium.

So, let us argue that $\Pr(I \subseteq cg) \rightarrow 1$. Since nobody ever bids in $I$, each bid is either up (above $\bar{I}$) or down (below $\tilde{I}$). The event $\{I \subseteq cg\}$ is equivalent to there being exactly $n$ ups. If there are $n + 1$ ups, then $I \subseteq lg$, and if there are $n - 1$, then $I \subseteq ug$. From the previous section, $\Pr(I \subseteq ug)$ and $\Pr(I \subseteq ug)$ do fall quickly. So it must be the case that the probability of exactly $n$ ups does not vanish quickly, but the probability of either $n - 1$ or $n + 1$ does vanish quickly.

We show that the only way for this to occur is if the system becomes essentially deterministic, and so $\Pr(I \subseteq cg) \rightarrow 1$.

To see how this works, let $p_i$ be the probability that $i$ bids up, and let $q_i$ be the probability he bids down. Order the players so that $p_i$ is decreasing. For any given realization, run along them and stop at the player $i$ when one counts $n - 1$ ups.

For $I \subseteq cg$, we need to hit exactly one more up in the rest of the sequence. If one hits no more ups, $I \subseteq ug$, while if one hits two more ups, $I \subseteq lg$, either of which is rare by Lemma 6. We argue that the only way to make one more up likely, but neither zero nor two more ups likely, is for the next player to have $p_{i+1}$ nearly 1 and for the remaining players to in aggregate have almost
no chance of even one up. Essentially, if \( p_{i+1} \) is not near 1, then, since \( p_i \) is decreasing, the probability on who is the \( n \)th up is spread out. However, then \( z \)-independence makes it likely that one also over- or undershoots by 1. Given that the next player is likely to hit, there must rarely be any more hits in the remaining population.

Running through the players in reverse order and counting downs, when one hits \( n - 1 \) downs, the next one must almost certainly play down and then there must almost never be any more downs. Since both of these are true at once, in aggregate, the first \( n \) bidders almost always bid up and the remaining down. Hence, \( \Pr(I \subseteq cg) \to 1 \).

The proof is distressingly long: \( cg \) can move around, sometimes including one interval and sometimes another, players might bid not only above or below any given \( I \), but sometimes within it, and one must be careful not to double count the ways in which a population “one player away” from creating a long \( cg \) might end up creating a long supporting gap. Most importantly, to derive the rate of convergence result, we need to consider situations in which \( I \) shrinks as \( n \) increases, and so we must be careful in accounting for the value of the occasional lost trade from bidding lower in an attempt to favorably affect the price.

4.5. Efficiency

We are now ready for our main theorem:

**Theorem 1:** All nontrivial equilibria of the single-unit demand/supply double auction are asymptotically efficient. Uniformly across nontrivial equilibria, efficiency losses go to zero faster than \( 1/n^{1-\alpha} \) for any given \( \alpha > 0 \). The fraction of expected surplus lost relative to a Walrasian market thus shrinks as \( 1/n^{2-\alpha} \).

For intuition, note that in Section 4.3 we showed that the efficiency loss from failing to trade objects between sellers with value below \( cg \) and buyers with values above \( \text{mg} \) is small (of order \( 1/n \)). So the only efficiency losses to worry about are from pairs of buyers and sellers both having value in \( cg \). The loss from missing such a trade is at most \( |cg| \), and, using NAA, the number of such buyers and sellers is like \( |cg|n \), so the deadweight loss triangle from too little trade has area \( |cg|^2n \). However, from Lemma 8, \( \Pr(|cg| \geq x) \leq 1/(n^{2-\alpha}x^2) \), and so the expected loss here is like \( 1/n \) as well. Finally, from NAG, expected feasible surplus grows like \( n \) and so proportional losses are like \( 1/n^2 \). A formal accounting of efficiency losses is subsumed by the proof of the multiple-unit case and so is omitted in the Appendix.

4.6. Asymptotic Uniqueness of Equilibrium

In the space of allocations, all nontrivial equilibria converge to the Walrasian outcome. Over “relevant” ranges, bids must thus converge to true values. So,
in the limit, if the Walrasian price is either \( p_1 \) or \( p_2 > p_1 \), then players with value near \( p_1 \) or \( p_2 \) must bid close to value. However, it is difficult to show that, for example, a player with value well above \( p_2 \) must bid near value. A rate of convergence result for bids is thus cumbersome. Intuitively, over relevant ranges, convergence should be order \( 1/n \).

5. MULTIPLE-UNIT DEMANDS AND SUPPLIES

Assume now that each player has demand or supply for at most \( m \) units for some fixed \( m \). For buyers, let \( v_{ih} \), \( h \in \{1, \ldots, m\} \), be \( i \)'s incremental value for unit \( h \).\(^{19}\) For sellers, let \( v_{ih} \) be the incremental cost of unit \( h \). We assume \( v_{ih} \) is nonincreasing in \( h \) for buyers and nondecreasing for sellers. Bids are (nonincreasing for buyers, nondecreasing for sellers) \( m \) vectors. Jackson and Swinkels (2005) applies to show existence of equilibria in this setting, subject to the same strengthenings as before.

We assume the following version of NAG.

**ASSUMPTION 5 —No Asymptotic Gaps\(^*\):** There is \( w > 0 \) such that for all \( n \), and for all intervals \( I \subseteq (0, 1) \) of length \( 1/n \) or greater,

\[
\sum_{i \in NB} P_i[v_{i1} \in I] \geq wn|I|
\]

and

\[
\sum_{i \in NS} P_i[v_{i1} \in I] \geq wn|I|.
\]

That is, when \( n \) is large, there are many buyers whose highest value might fall in any given interval and many sellers whose lowest cost might fall into any given interval.\(^{20}\)

Note that \( z \)-independence applies only across players and thus does not restrict the relationship of the different values of any given player. The NAA condition is assumed to apply to all values, not just the first, so not too many \( v_{ih} \) fall in any given interval.

**THEOREM 2:** With NAG\(^*\), Theorem 1 continues to hold even with multiple-unit demands and supplies.

\(^{19}\)As before, we include atoms for buyers at 0 and sellers at 1, so this does not imply that buyers have positive value for all \( m \) units or that sellers want or are able to sell \( m \) units.

\(^{20}\)There are less restrictive ways in which one might generalize NAG. For example, if each buyer’s first value is uniform \([3, 4]\), and his second value is uniform \([0, v_{i1}]\), then there are many buyer values in each range. An example in Section 5.1 of Swinkels (2001) suggests that this is not strong enough to guarantee efficiency.
Most of the incentive arguments rely only on the highest value unit of demand for buyers and lowest cost unit for sellers. The proof proceeds in two steps. Define \( u_g \) as the \( m \)th bid up from \( cg \) and let \( ug \equiv [cg, u_g] \). In the Appendix, we show that Lemma 6 continues to hold for this definition of \( ug \). The modification to the intuition is very small: when \( ug \) is long, there are many buyers with highest value in the top half of \( ug \), but only \( m \) of them can be winning a first object. Given this, Lemma 8 is easily extended as well. Instead of sorting players into those who play up and down, sort them into those who make zero up bids, one up bid, etc. This is notationally intensive, but straightforward and hence omitted.

Finally, we must show that since \(|ug|, |cg|, \) and \(|lg|\) shrink quickly, inefficiency in the market disappears as \( 1/n \). A proof of this is in the Appendix. To see the issues involved, note that for the single-unit case (and for the first unit of demand in the multiple-unit case), a buyer’s impact on the price is small for two reasons. First, he is unlikely to be pivotal. Second, even if he is pivotal, he does not affect the price much, since the next bid up is likely to be close. We exploit both of these forces in showing Lemmas 6 and 8 and their adaptations here.

For units of demand after their first, many buyers can simultaneously be in the position that in raising bids other than their first, they pay more for units they were already winning. To get around this, consider the deviation to honest bidding. In any given realization, let \( x \) be \( u_g - cg \). This is the maximum impact of \( i \) raising his \( m \) bids on price. If \( vih < cg \), then the deviation is irrelevant. If \( cg \leq vih \leq cg + 2mx \), then \( i \) may not benefit very much from any new unit won by raising \( bih \) and may hurt himself by raising the price by as much as \( x \) on each of \( m - 1 \) units already being won. However, critically, because of NAA, the number of \( vih \) in \((cg, cg + 2mx)\) is only like \( nx \) (as always, ignoring constants), so the expected cost to bidders from this case is like \( E(nx^2) \). However, the modified versions of Lemmas 6 and 8 give that \( E(nx^2) \) is like \( 1/n \),\(^{21}\) and the expected efficiency loss from such players not winning also falls like \( 1/n \).

Consider objects with \( vih \) above \( cg + 2mx \), where \( i \) is already winning an \( h \)th object. As before, for only one player can one of the associated bids be \( cg \), so the sum of costs in terms of raising these bids is at most \( x \) and \( E(x) \leq 1/n \) as well.

The remaining objects have \( vih \) above \( cg + 2mx \) but are not winning. The deviation to \( v \) thus wins an extra object at price at most \( cg + x \) and raises the price by at most \( x \) on \( m - 1 \) units, for a net profit of \( vih - cg - mx \). The efficiency loss from \( i \) not winning object \( h \) is at most \( vih - cg \), which, given that \( vih - cg > 2mx \), is at most twice \( vih - cg - x \). So on these objects, bidder’s

\(^{21}\)The lower bound on the distribution function of \( x \) will be used to bound its expectation in our proof of Theorem 2.
profits from the deviation are at least half of the efficiency loss on these units. Since costs from raising bids on other units are insignificant, it follows that the efficiency loss on these units is small, since otherwise bidders will in aggregate have a profitable deviation. Since the efficiency loss on other units is also small, we are done.

6. EXTENSIONS

6.1. One-Sided Uniform-Price Auctions

Swinkels (2001) considers large one-sided auctions with independent values and a little bit of “noise.” An example is if there is a small independent probability that each player sleeps through the auction. In the uniform price case, it is shown that with the noise, the impact that any given player has on the price grows small in expectation. Then, since “honest” bidding has a small effect on the price paid, it must also have little benefit in winning extra objects. This implies asymptotic efficiency (without a rate of convergence).

An easy extension to the arguments here shows that a one-sided uniform price auction with $z$-independent values converges to efficiency at rate $1/n^{2-\alpha}$, even without noise. This paper thus significantly generalizes Swinkels (2001) for the uniform price case. The key is that here we think of “cost” as the impact on price in circumstances where the player affecting the price cares. This is a simpler object to bound, allowing both the greater generality and fast convergence.

6.2. Weaker Information Assumptions

We can weaken the information assumptions considerably and still obtain convergence. There is no problem if most players have considerably more knowledge about each other’s values than $z$-independence allows. What counts (for convergence) is that from the point of view of a nonvanishing fraction of players, there are “lots” of players who cannot be predicted precisely and that NAG applies to this set of players. Depending on the details these weakenings may or may not permit a result on rates of convergence.

6.3. Nonprivate Values

Assume that an $\varepsilon$ fraction of the players have pure private values, while the remaining players have values with a common component. Much as before, players with private values will bid close to value. By NAG, their bids are

\[ 22 \text{The stronger notion of vanishing impact is needed to prove results for discriminatory auctions, which are also analyzed in that paper.} \]
closely packed almost surely, and thus the impact of bids on price disappears for all players. It is an interesting question under what conditions this implies an information aggregation and efficiency result. The key difficulty is ruling out pooling (or partial pooling) among those players whose values have a common component.

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APPENDIX: PROOFS

PROOF OF LEMMA 1: Without loss of generality, let $K = \{1, 2, \ldots, |K|\}$. Let $P_K$ and $P_{N\setminus K}$ be the marginals of $P$ on $K$ and $N \setminus K$, respectively, and let $P_{K \times N \setminus K}$ be the associated product measure. Fix a rectangular event $F_K = F_1 \cap F_2 \cap \cdots \cap F_{|K|}$, where each $F_i$ only involves $v_i$. Then

$$\Pr(F_K | F_{N \setminus K}) = \prod_{i=1}^{|K|} \Pr(F_i | F_{i+1} \cap \cdots \cap F_{|K|} \cap F_{N \setminus K})$$

$$\leq z^{-|K|} \prod_{i=1}^{|K|} \Pr(F_i | F_{i+1} \cap \cdots \cap F_{|K|}) \quad \text{(using } z\text{-independence})$$

$$= z^{-|K|} \Pr(F_K) = z^{-|K|} P_K(F_K).$$

Analogously,

$$\Pr(F_K | F_{N \setminus K}) \geq z^{|K|} P_K(F_K).$$

These inequalities extend to any $F_K$ in the product $\sigma$-algebra, because such a set is the limit of a countable union of rectangles. Thus

(A.1) $z^{-|K|} \Pr(F_K) \Pr(F_{N \setminus K}) \geq \Pr(F_K \cap F_{N \setminus K}) \geq z^{|K|} \Pr(F_K) \Pr(F_{N \setminus K})$

or, equivalently,

(A.2) $z^{-|K|} P_{K \times N \setminus K} \geq P \geq z^{|K|} P_{K \times N \setminus K}$. 

Let $F_K$ and $F_{N\setminus K}$ be events about values and bids. Then

$$\Pr(F_K \cap F_{N\setminus K})$$

$$= \int_{[0,1]^{|N|}} \Pr(F_K | v_K) \Pr(F_{N\setminus K} | v_{N\setminus K}) dP$$

$$\leq z^{-|K|} \int_{[0,1]^{|K|}} \Pr(F_K | v_K) dP_K \int_{[0,1]^{|N|-|K|}} \Pr(F_{N\setminus K} | v_{N\setminus K}) dP_{N\setminus K}$$

$$= z^{-|K|} \Pr(F_K) \Pr(F_{N\setminus K}).$$

The product in the first integral is defined by the players’ distributional strategies; the second line uses (A.2); the third line applies Fubini’s theorem; the final line integrates. Similarly $\Pr(F_K \cap F_{N\setminus K}) \geq z^{-|K|} \Pr(F_K) \Pr(F_{N\setminus K})$ and so (A.1) holds for all events.

Similarly, for rectangular events $F_{N\setminus K}$,

(A.3) \[ z^{-|N\setminus K|} \Pr(F_K) \Pr(F_{N\setminus K}) \geq \Pr(F_K \cap F_{N\setminus K}) \geq z^{-|N\setminus K|} \Pr(F_K) \Pr(F_{N\setminus K}). \]

Exchanging $K$ and $N \setminus K$ in the above and combining yields

(A.4) \[ z^{-a} \Pr(F_K) \Pr(F_{N\setminus K}) \geq \Pr(F_K \cap F_{N\setminus K}) \geq z^{-a} \Pr(F_K) \Pr(F_{N\setminus K}). \]

Dividing through by $\Pr(F_{N\setminus K})$ gives (2).

Let $X_K$ be a step function with values $x^a$ on a finite partition $\{F^a\}_{a \in A}$, where each $F^a$ is an event on bids and values in $K$. By the definition of conditional expectation, $E(X_K | F_{N\setminus K}) = \sum_{a \in A} x^a \Pr(F^a | F_{N\setminus K})$. Thus by (2),

$$E(X_K | F_{N\setminus K}) \leq z^{-a} \sum_{a \in A} x^a \Pr(F^a) = z^{-a} E(X_K).$$

Analogously, $E(X_K | F_{N\setminus K}) \geq z^{-a} E(X_K)$. Since an arbitrary $X_K$ is the limit of such step functions, (3) follows.

Q.E.D.

**Proof of Lemma 2:** Without loss of generality, let $K = \{1, 2, \ldots, \kappa\}$. Define the Bernoulli process with $\kappa$ independent trials with success probability $\bar{p}_i$ in trial $i$. Let $x_i \in \{0, 1\}$ be the outcome of trial $i$ and let $X^k = \sum_{i=1}^k x_i$. We claim that $X^k \equiv X^x$ FOSD $Q_K$ given $F_{N\setminus K}$. The proof is inductive. Let $Q^K$ be the number of $F_1, \ldots, F_k$ that occur. Trivially, $X^0$ FOSD $Q^0$, since both are
identically 0. Suppose $X^{k-1}$ FOSD $Q^{k-1}$ given $F_{N\setminus K}$. Then, for $r \in \{0, \ldots, k\}$,

$$
\Pr(Q^k \leq r \mid F_{N\setminus K}) = \Pr(Q^{k-1} < r \mid F_{N\setminus K}) + \Pr(F^c_k \mid \{Q^{k-1} = r\} \cap F_{N\setminus K}) \Pr(Q^{k-1} = r \mid F_{N\setminus K})
$$

$$
\geq \Pr(Q^{k-1} < r \mid F_{N\setminus K}) + (1 - \overline{p}_k) \Pr(Q^{k-1} = r \mid F_{N\setminus K})
$$

$$
= \overline{p}_k \Pr(Q^{k-1} < r \mid F_{N\setminus K}) + (1 - \overline{p}_k) \Pr(Q^{k-1} \leq r \mid F_{N\setminus K})
$$

$$
\geq \overline{p}_k \Pr(X^{k-1} < r) + (1 - \overline{p}_k) \Pr(X^{k-1} \leq r)
$$

$$
= \Pr(X^k \leq r).
$$

The first inequality uses the definition of $\overline{p}_k$ and (4). The middle equality uses $\Pr(Q^{k-1} = r) = \Pr(Q^{k-1} \leq r) - \Pr(Q^{k-1} < r)$ and the final inequality uses the inductive hypothesis.

Similarly, if $Y_K$ is the number of successes in a Bernoulli process with success probabilities $p_i$, then given $F_{N\setminus K}$, $Q_K$ FOSD $Y_K$.

We want a large-deviations inequality for the bounding Bernoulli processes. Since $X_K$ is a sum of nonidentical independent Bernoulli trials, a slight alteration to the usual proof of Cramér’s theorem (e.g., Shiryaev (1996, p. 68)) is necessary. Let $\pi = \frac{1}{\kappa} \sum_{i} p_i$. Then, for any $\phi \geq 0$ and $\lambda > 0$,

$$
\Pr\left(\frac{X_K}{\kappa \pi} > \phi\right) = \Pr(e^{\lambda X_K/\kappa \pi} \geq e^{\lambda \phi}) \leq \frac{E(e^{\lambda X_K/\kappa \pi})}{e^{\lambda \phi}}
$$

by Markov’s inequality. Note also that $\Pr(X_K/(\kappa \pi) > \phi) = 0$ trivially when $\pi \phi \geq 1$. So assume that $\pi \phi < 1$.

Because $X_K$ is a sum of independent random variables,

$$
Ee^{\lambda X_K/\kappa \pi} = \prod_{i \in K} (1 - \overline{p}_i + \overline{p}_i e^{\lambda/\kappa \pi})
$$

$$
= \exp\left(\log \prod_{i \in K} (1 - \overline{p}_i + \overline{p}_i e^{\lambda/\kappa \pi})\right)
$$

$$
= \exp\left(\sum_{i \in K} \log(1 - \overline{p}_i + \overline{p}_i e^{\lambda/\kappa \pi})\right)
$$

$$
\leq \exp(\kappa \log(1 - \pi + \pi e^{\lambda/\kappa \pi}))
$$

since $\log(1 - x + xe^{\lambda/\kappa \pi})$ is concave in $x$. 


Thus,

\[
\Pr\left( \frac{X_K}{\kappa \pi} > \phi \right) \leq \exp \left[ -\lambda \phi + \kappa \log(1 - \pi + \pi e^{\lambda/\kappa \pi}) \right]
\]

\[
= \exp \left[ -\kappa \left\{ \frac{\lambda}{\kappa \pi} \phi \pi - \log(1 - \pi + \pi e^{\lambda/\kappa \pi}) \right\} \right]
\]

\[
= \exp \left[ -\kappa \{s \phi \pi - \log(1 - \pi + \pi e^s)\} \right],
\]

where \( s \equiv \frac{s}{s} \). Given that \( \lambda > 0 \) was arbitrary, this holds for all \( s > 0 \), and so in particular for \( s = \log\left(\frac{(1 - \pi)\phi}{1 - \pi\phi}\right) \) (this is positive, because \( \pi\phi < 1 \)), yielding

\[
\Pr\left( \frac{X_K}{\kappa \pi} > \phi \right) \leq \exp \left[ -\kappa \phi \pi \log(1 - \pi + \pi e^{s}) \right].
\]

Since \( \log x \geq (x - 1)/x \), the second term in the braces is at least \( \pi(1 - \phi) \). Thus,

\[
\Pr\left( \frac{X_K}{\kappa \pi} > \phi \right) \leq \exp[ -\kappa \pi(\phi \log \phi + 1 - \phi)].
\]

Choosing \( \phi = 3 \),

\[
\Pr(X_K > 3\kappa \pi) \leq e^{-\kappa \pi(3 \log 3 - 2)} \leq e^{-\kappa \pi}.
\]

Note that \( \frac{1}{2}E(Q_K) \geq \sum_{i \in K} Q_i = \kappa \pi \); hence, \( \Pr(X_K > 3\kappa \pi) \geq \Pr(X_K > \frac{3}{2}E(Q_K)) \). Note also that \( \sum_{i \in K} \overline{P}_i \geq E(Q_K) \), and so \( e^{-\kappa \pi} \leq e^{-E(Q_K)} \). Finally, \( X_K \) stochastically dominates \( Q_K \). Taken together with (A.8), this implies

\[
\Pr\left( Q_K > \frac{3}{2}E(Q_K) \right) \leq e^{-E(Q_K)},
\]

giving (6).

The proof for \( Y_K \) is similar: Define \( \pi = \sum_{i \in K} \overline{P}_i / \kappa \pi \). Then, for any \( \lambda < 0 \) and \( 0 < \phi < 1 \),

\[
\Pr\left( \frac{Y_K}{\kappa \pi} < \phi \right) = \Pr(e^{\lambda X_K/\kappa \pi} \geq e^{\lambda \phi}) \leq \frac{E(e^{\lambda X_K/\kappa \pi})}{e^{\lambda \phi}}.
\]
The derivation of (A.5) and (A.6) is then as before, replacing \( p_i \) by \( p_i^* \) and \( \Pr(X_K / (\kappa \pi) > \phi) \) by \( \Pr(Y_K / (\kappa \pi) < \phi) \). Note in particular that since \( \lambda < 0 \), \( s \equiv \frac{1}{\kappa \pi} \) can once again take on any positive value. Setting \( s = \log(\frac{1-\pi}{1-\pi\phi}) \) is once again valid, because \( \phi < 1 \). Hence we arrive at the analogue to (A.7):

\[
(A.9) \quad \Pr(\frac{Y_K}{\kappa \pi} < \phi) \leq \exp[-\kappa \pi(\phi \log \phi + 1 - \phi)].
\]

Note that \( \sum_{i \in K} p_i \geq zE(Q K) \), so that \( \Pr(Y_K < \phi E(Q K)) \leq \Pr(Y_K < \phi \kappa \pi) \) and \( \exp[-\kappa \pi(\phi \log \phi + 1 - \phi)] \leq \exp[-zE(Q_K)(\phi \log \phi + 1 - \phi)] \). So

\[
(A.10) \quad \Pr(Q_K < \phi zE(Q K)) \leq \exp(-zE(Q K)(\phi \log \phi + 1 - \phi)).
\]

Since \( \frac{1}{3} \log \frac{1}{3} + 1 - \frac{1}{3} > 0.3 \), (5) follows. \( Q.E.D. \)

**Proof of Corollary 1:** Note that \( \Pr(Q_K = 0 | F_{N\setminus K}) \leq \Pr(Q_K \leq \phi E(Q K)) \) for any \( \phi > 0 \). Equation (A.10) then gives

\[
\Pr(Q_K = 0 | F_{N\setminus K}) \leq \exp(-zE(Q_K)(\phi \log \phi + 1 - \phi))
\]

since \( E(Q_K) \geq \Pr(Q_K \geq 1) \). Since this holds for \( \phi \) arbitrarily close to 0 and as \( \lim_{\phi \to 0} \phi \log \phi = 0 \),

\[
\Pr(Q_K \geq 1 | F_{N\setminus K}) \geq 1 - e^{-z\Pr(Q_K \geq 1)}
\]

\[
= 1 - \frac{e^{-z\Pr(Q_K \geq 1)}}{\Pr(Q_K \geq 1)} \Pr(Q_K \geq 1).
\]

For \( x \in (0, 1] \), \( (1 - e^{-zx})/x \) is minimized at \( x = 1 \). \( Q.E.D. \)

**Proof of Lemma 3:** Let \( [x] \) denote the integer part of \( x \). Partition \([0, 1] \) into \( k \equiv [n^{1-a/4}] \) intervals \( I_k \) of equal length (between \( 1/n1-a/4 \) and \( 1/(2n1-a/4) \)). Let \( Q_B(I_k) \) be the number of buyers with values in \( I_k \). Note that

\[
Wn/k = Wn|I_k| \geq E(Q_B(I_k)) \geq wn|I_k| = wn/k.
\]

Let \( E_{1k} \equiv \{ \frac{3}{2} Wn/k \geq Q_B(I_k) \geq \frac{1}{3} wn/k \} \). By Lemma 2,

\[
\Pr(E_{1k}) \geq 1 - 2e^{-0.32wn/k} \geq 1 - \frac{1}{n^3}
\]

for \( n \) SL, since \( n/k \to n^{a/4} \). Similarly, let \( Q_s(I_k) \) be the number of sellers with values in \( I_k \) and define \( E_{2k} \equiv \{ \frac{3}{2} Wn/k \geq Q_s(I) \geq \frac{1}{3} wn/k \} \). Then \( \Pr(E_{2k}) \geq 1 - 1/n^5 \) for \( n \) SL.
Let $\hat{N} \equiv \bigcap_{\kappa} (E_{1\kappa} \cap E_{2\kappa})$. Then, for $n$ SL,

(A.11) \[ \Pr(\hat{N}) \geq 1 - \sum_{\kappa} \Pr(E_{1\kappa}^c) - \sum_{\kappa} \Pr(E_{2\kappa}^c) \]

(A.12) \[ \geq 1 - 2 \frac{n^{1-\alpha/4}}{n^5} \]

\[ \geq 1 - 1/n^4. \]

The first inequality holds because we are just double counting times where $\hat{N}$ does not hold.

Finally, note that for $n$ SL, any interval $I$ of length at least $1/n^{1-\alpha/3}$ contains at least $k|I| - 2 \geq k|I|/2$ elements of $\{I_\kappa\}$. So, given $\hat{N}$,

$$QB(I) \geq \frac{k|I| w n}{2z} = \frac{nzw}{6}|I|.$$  

Similarly, $I$ intersects with at most $2k|I|$ elements of $\{I_\kappa\}$ and so $QB(I) \leq \frac{5}{2}Wn|I|$. The argument for $QS(I)$ is analogous. Thus, in fact $\hat{N} \subset N$ and we are done. 

Q.E.D.

PROOF OF PROPOSITION 1: To prove Proposition 1 we need a technical lemma.

LEMMA 9: Let a subsequence of auctions $\{A^n\}_{n=0}^\infty$ and associated equilibria be given. If $(E(V | T))/n_t \geq \gamma$ for some $\gamma > 0$ and all $t$, then $Pr(T) \rightarrow 1$ along this subsequence.

Intuitively, if many players trade given $T$, then many players must occasionally be bidding in a fairly aggressive way. Then, by $z$-independence, at least a fraction of them will be doing so almost all the time. The proof is more complicated because the event $T$ is linked to all players’ actions and so $z$-independence does not immediately apply.

PROOF OF LEMMA 9: Choose a subsequence $\{n_t\}$ such that $E(V | T) \geq n_t \gamma$ for all $t$. (We will suppress the subscript in what follows.) Let $\gamma' = \gamma/2$. Then, since $E(V | T) \geq n \gamma$, $Pr(V > \gamma' n | T) > \gamma'$ since $V \leq n$. Given the event $\{V > \gamma' n\}$, if one selects $\gamma' n/2$ of the buyers at random, the probability that none trades is at most $(1 - \gamma')^{\gamma' n/2} \leq 1/8$ for $n$ SL, and so there is a $7/8$ probability that at least one such buyer is trading. Since this is true in expectation, it must be true for some particular set $G_B$ of $\gamma' n/2$ buyers. Similarly, there is a set $G_S$ of $\gamma' n/2$ sellers such that, conditional on $\{V > \gamma' n\}$, at least one seller in $G_S$ is a trader with probability $7/8$. Let $G \equiv G_S \cup G_B$ and let $T_G \subset T$ be the event that at least
one buyer in \( G \) trades and one seller in \( G \) trades. Then \( \Pr(T_G \mid \{V > \gamma' n\}) \geq 1 - 2(1/8) = 3/4 \) and so

\[
\text{(A.13)} \quad \Pr(T_G \cap \{V > \gamma' n\}) \geq \frac{3}{4} \Pr(V > \gamma' n).
\]

Thus

\[
\Pr(\{V > \gamma' n\} \mid T_G) = \frac{\Pr(T_G \cap \{V > \gamma' n\})}{\Pr(T_G)} \geq \frac{3}{4} \frac{\Pr(V > \gamma' n)}{\Pr(T)} \geq \frac{3 \gamma'}{4},
\]

where the first inequality uses (A.13) and \( T_G \subset T \), while the second uses \( \Pr(V > \gamma' n) > \gamma' \).

Let \( X \) be the event that at least \( \frac{\gamma' n}{2} \) buyers and \( \frac{\gamma' n}{2} \) sellers in \( N \setminus G \) trade. Since \( G \) has only \( \frac{\gamma' n}{2} \) buyers or sellers, \( T_G \cap \{V > \gamma' n\} \) implies \( X \), so \( \Pr(X \mid T_G) \geq \frac{3}{4} \).

Let \( p^* \) be such that \( \Pr(p \geq p^\mid X \cap T_G) \geq \frac{1}{2} \) and \( \Pr(p \leq p^\mid X \cap T_G) \geq \frac{1}{2} \).

Let \( Q_S \) be the number of sellers in \( N \setminus G \) with \( b_i \leq p^* \) and let \( Q_B \) be the number of buyers in \( N \setminus G \) with \( b_i \geq p^* \). Then

\[
E(Q_S \mid T_G) \geq \Pr(X \cap \{p \leq p^*\} \mid T_G) \frac{\gamma' n}{2} \geq \frac{1}{2} \Pr(X \mid T_G) \frac{\gamma' n}{2} \geq \frac{3}{16} \gamma^2 n
\]

and so

\[
E(Q_S) = \sum_{i \in N_S \setminus G} \Pr(b_i \leq p^*) \geq z \sum_{i \in N_S \setminus G} \Pr(b_i \leq p^* \mid T_G) = zE(Q_S \mid T_G) = 3z \gamma^2 n/16.
\]

Thus, by Lemma 2, \( \Pr(Q_S = 0) \to 0 \) as \( n \to \infty \). Similarly \( \Pr(Q_B = 0) \to 0 \). Thus, along this subsequence, \( \Pr(Q_S > 0 \text{ and } Q_B > 0) \to 1 \). When \( Q_S > 0 \) and \( Q_B > 0 \) there is at least one buy bid above a sell bid, so \( \Pr(T) \to 1 \). 

Q.E.D.

This in hand, fix \( A^n \) and a nontrivial equilibrium. Let \( \phi_B \equiv \max_{N_H} b_i \) be the highest buy bid submitted and let \( \phi_S \equiv \min_{N_S} b_i \) be the lowest sell bid. Note that \( \Pr(\phi_B \geq x) \) is decreasing and continuous from the left. Similarly, \( \Pr(\phi_S \leq x) \) is increasing and continuous from the right. Let \( v^* \in [0, 1] \) have the property that
for all $x \in [0, v^*)$, $\Pr(\phi_B \geq x) \geq \Pr(\phi_S \leq x)$, while for all $x \in (v^*, 1]$, $\Pr(\phi_B \geq x) \leq \Pr(\phi_S \leq x)$. Let
\[
\delta \equiv \min\{\Pr(\phi_B \geq v^*), \Pr(\phi_S \leq v^*)\}.
\]
Note that $\Pr(\phi_B > v^*) \leq \delta$. This is trivial if $\Pr(\phi_B \geq v^*) = \delta$. If $\Pr(\phi_B \geq v^*) > \delta$, then $\Pr(\phi_S \leq v^*) = \delta$. Then since $\Pr(\phi_S \leq x)$ is continuous from the right,
\[
\Pr(\phi_B > v^*) = \lim_{v \downarrow v^*} \Pr(\phi_B \geq v) \leq \lim_{v \downarrow v^*} \Pr(\phi_S \leq v) = \Pr(\phi_S \leq v) = \delta.
\]
Analogously, $\Pr(\phi_S < v^*) \leq \delta$.

Assume that $\Pr(\phi_S \leq v^*) = \delta$. Then $\Pr(T \cap \{p \leq v^*\}) \leq \Pr(\phi_S \leq v^*) = \delta$, while $\Pr(T \cap \{p > v^*\}) \leq \Pr(\phi_B > v^*) \leq \delta$. Similarly, if $\Pr(\phi_B \geq v^*) = \delta$, then $\Pr(T \cap \{p < v^*\}) \leq \Pr(\phi_S < v^*) \leq \delta$, while $\Pr(T \cap \{p \geq v^*\}) \leq \Pr(\phi_B \geq v^*) = \delta$, so $\Pr(T) \leq 2\delta$.

Now $\{\phi_S \leq v^*\} = \bigcup_{i \in N_S} \{b_i \leq v^*\}$. Hence, by Corollary 1,
\[
(A.14) \quad \Pr(\phi_S \leq v^* | F_{N_B}) \geq (1 - e^{-z}) \Pr(\phi_S \leq v^*)
\]
for any $F_{N_B}$, so
\[
(A.15) \quad \Pr(T) \geq \Pr(\{\phi_B \geq v^*\} \cap \{\phi_S \leq v^*\})
\]
\[
\geq \Pr(\phi_S \leq v^* | \phi_B \geq v^*) \Pr(\phi_B \geq v^*)
\]
\[
\geq (1 - e^{-z}) \delta^2.
\]

Assume that $v^* \leq 1/2$. (If not, the proof below applies, \textit{mutatis mutandis}, to the sellers.) Fix an arbitrary buyer $i$. Let $\phi_B^i \equiv \max_{N_B \setminus \{i\}} b_i$. Now
\[
\Pr(\phi_B^i < 2/3) = 1 - \Pr(\phi_B^i \geq 2/3)
\]
\[
\geq 1 - \Pr(\phi_B > v^*)
\]
\[
\geq 1 - \delta.
\]

Let $J \equiv [5/6, 1]$. By Lemma 1,
\[
(A.16) \quad \Pr(\phi_B^i < 2/3 | v_i \in J) \geq z(1 - \delta).
\]

By (A.14),
\[
(A.17) \quad \Pr(\phi_S \leq v^* | v_i \in J, \phi_B^i < 2/3) \geq (1 - e^{-z}) \delta.
\]
Let $d_i$ be the deviation for $i$ that whenever $v_i \in J$ and the original strategy specified a bid below $v^*$, he bids $2/3$ instead. Under this strategy, he wins an object with probability at least $\Pr(\phi_B < 2/3, \phi_S \leq v^*, v_i \in J)$, which by (A.16) and (A.17) is at least

$$\Pr(v_i \in J) z (1 - \delta) (1 - e^{-z}) \delta,$$

and earns at least $\frac{1}{6}$ when he does so. So

$$B_i \geq \Pr(v_i \in J) z (1 - \delta) (1 - e^{-z}) \delta \frac{1}{6} - \pi_i,$$

where $\pi_i$ is $i$’s expected equilibrium profit.

Summing across buyers and applying Lemma 4 yields

(A.18) \[ z (1 - \delta) (1 - e^{-z}) \delta \frac{1}{6} \sum_{N_B} \Pr(v_i \in J) - \sum_{N_B} \pi_i \leq \Pr(T). \]

By Assumption 2, $\sum_{N_B} \Pr(v_i \in J) \geq \frac{1}{6} w n$ for $n \geq 6$. Whereas the gains to a buyer from any given trade are at most 1 and $V$ buyers trade,

$$\sum_{N_B} \pi_i \leq \Pr(T) E(V \mid T).$$

Substituting into (A.18) gives

$$z (1 - \delta) (1 - e^{-z}) \delta \left( \frac{1}{6} \right)^2 w n - \Pr(T) E(V \mid T) \leq \Pr(T).$$

Using $\Pr(T) \leq 2\delta$, dividing through by $2n\delta > 0$, and rearranging gives

$$\frac{1}{72} w z (1 - \delta) (1 - e^{-z}) - \frac{1}{n} \leq \frac{E(V \mid T)}{n}.$$

Suppose that the proposition were false. Then there exists a subsequence $\{n_t\}$ such that $\Pr(T) \to 0$ as $t \to \infty$. Along this subsequence $\delta \to 0$, because $\Pr(T) \geq (1 - e^{-z}) \delta^2$ (by (A.15)). Hence, along this subsequence the left-hand side above converges to $w z \gamma (1 - e^{-z}) / 72 > 0$. By Lemma 9, therefore, $\Pr(T) \to 1$ along this subsequence—a contradiction. \textit{Q.E.D.}

\textbf{PROOF OF LEMMA 5:} We will prove stronger results that will be useful when we turn to the multiple-unit case. Fix an integer $m \geq 1$. Redefine $\overline{ug}$ as the $m$th bid above $\overline{cg}$. As before, let $ug \equiv (\overline{cg}, \overline{ug})$. When $m = 1$, we have the original case.
Let us show that for \( n \) SL, \( E(|ug|) \leq 1/n^{1-\alpha/4} \). Assume this is false along a subsequence. Let \( L \equiv N \cap \{ |ug| > \frac{1}{2} E(|ug|) \} \). Then, since \( |ug| \leq 1 \),

\[
    E(|ug|) \leq \Pr(N)E(|ug| | N) + \frac{1}{n^4} \quad \text{(for } n \text{ SL)}
    \]

\[
    = \Pr(L)E(|ug| | L) + \Pr(N \setminus L)E(|ug| | N \setminus L) + \frac{1}{n^4}
    \]

So, for \( n \) SL,

\[
    (A.19) \quad \Pr(L)E(|ug| | L) > \frac{E(|ug|)}{3}.
    \]

Consider \( d_i(b_i, v_i) = \max\{b_i, v_i - \frac{E(|ug|)}{4} \} \) and assume \( L \) holds. Since \( |ug| > E(|ug|)/2 > E(|ug|)/4 > 1/(4n^{1-\alpha}) > 1/n^{1-\alpha/3} \) for \( n \) SL and given that \( L \) holds, the number of new winners is at least \( w' \frac{1}{4} n|ug| - m \geq \frac{w'}{8} n|ug| \) for \( n \) SL. Each new winner earns at least \( \frac{1}{4} E(|ug|) \). So, using (A.19),

\[
    \sum B_i \geq \frac{E(|ug|)}{4} \frac{w'}{8} nE(|ug| | L) \Pr(L)
    \]

\[
    \geq \frac{E(|ug|)}{4} \frac{w'}{8} n \frac{E(|ug|)}{3} = \frac{E(|ug|)^2}{96} w'n.
    \]

However, by Lemma 4, \( \sum B_i \leq E(|ug|) \) and so

\[
    \frac{E(|ug|)^2}{96} w'n \leq E(|ug|)
    \]

or

\[
    nE(|ug|) \leq \frac{96}{w'}.
    \]

For \( n \) SL, this contradicts \( E(|ug|) \geq 1/n^{1-\alpha/4} \). The argument for sellers is analogous. \( Q.E.D. \)

**Proof of Lemma 6:** We proceed by contradiction. Note first that if \( x \leq 1/n^{1-\alpha_3/2} \), then \( 1/(nx^2n^{1-\alpha_3}) \geq 1 \) and the claim is vacuous. So, along some
subsequence \( \{n_t, x_t\}, n_t \to \infty \), assume \( \Pr(\{|ug| > x_t\}) > \frac{1}{(x_t^2 n_t^{1-a_3})} \) and \( x_t > \frac{1}{n_t^{1-a_3/2}} \). Then, for \( n \) SL,

\[
\Pr(N \cap \{|ug| > x\}) > \Pr(|ug| > x) - \frac{1}{n^4} > \frac{\Pr(|ug| > x)}{2}
\]

(omitting \( t \)). Consider \( d_i(b, v_i) = \max\{b_i, v_i - \frac{x}{2}\} \). Now consider the event \( N \cap \{|ug| > x\} \). Under normality the number of buyers in the top half of \( ug \) is at least \( w'nx/2 \) (note that \( x/2 > 1/(2n^{1-a_3/2}) > 1/n^{1-a_3} \) for \( n \) SL, so Definition 2 does apply). So there are \( w'nx/2 - m > w'nx/3 \) new winners, each earning \( x/2 \). So

\[
\sum_{N_S} B_i \geq \Pr(|ug| > x) \frac{w'nx^2}{6}.
\]

However, using Lemmas 4 and 5, \( \sum B_i \leq 1/n^{1-a_4} \). So

\[
\Pr(|ug| > x) \frac{w'nx^2}{6} \leq \frac{1}{n^{1-a_4}}.
\]

Rearranging yields

\[
\Pr(|ug| > x) \leq \frac{6}{wn^{2-a_4}}.
\]

For \( n \) SL, this contradicts \( \Pr(|ug| > x) > 1/(x^2 n^{2-a_3}) \). \( \) Q.E.D. \( \)

\textbf{Proof of Lemma 7}: For buyer \( i \), consider \( d_i(b, v_i) = \max\{b_i, v_i - \frac{x}{2}\} \). If \( i \in L_B(x) \), then \( i \) wins an extra object and earns at least \( x \). By Lemmas 4 and 5,

\[
x E(l_B(x)) \leq \sum B_i \leq E(|ug|) \leq \frac{1}{n^{1-a_4}}
\]

and so

\[
\text{(A.20)} \quad E(l_B(x)) \leq \frac{1}{xn^{1-a_4}},
\]

establishing the first claim. Now note that in each realization

\[
Y_B \left( \frac{1}{n} \right) = \frac{1}{n} l_B \left( \frac{1}{n} \right) + \int_{1/n}^1 l_B(x) \, dx.
\]

(The first term is the rectangle and the second term is the “triangle” in a consumer surplus calculation for demand curve \( l_B(\cdot) \) up to demand \( Q = l_B(1/n) \).)
Therefore, by Fubini’s theorem,
\[
E \left( Y_B \left( \frac{1}{n} \right) \right) = \frac{1}{n} E \left( l_B \left( \frac{1}{n} \right) \right) + \int_{1/n}^{1} E \left( l_B (x) \right) dx \\
\leq \frac{1}{n^{1-\alpha}} + \int_{1/n}^{1} \frac{1}{x n^{1-\alpha}} dx \quad \text{(using (A.20) twice)} \\
= \frac{1}{n^{1-\alpha}} \left( 1 + \log n \right) \\
\leq \frac{1}{n^{1-\alpha_3}} \quad \text{(for } n \text{ SL)},
\]
which establishes the second claim. Repeat for sellers. \(Q.E.D.\)

**PROOF OF LEMMA 8:** Suppose the lemma is false, so that there exists a sequence \(\{n_i\}, \{x_i\}\) satisfying \(n_i \to \infty\) and
\[
(\text{A.21}) \quad x_i \geq 1/n_i^{1-\alpha/2}
\]
such that \(\Pr(|cg| \geq x_i) \geq 1/(n_i^{2-\alpha}x_i^2)\) (the claim is vacuous for smaller \(x_i\)).

**Step 1—Sparse Intervals:** Recall from the proof of Lemma 3 the partition of \([0, 1]\) into \(k \equiv [n_i^{1-\alpha/4}]\) disjoint intervals \(\{I_\kappa\}\) of equal length between \(1/n_i^{1-\alpha/4}\) and \(2/n_i^{1-\alpha/4}\). Let \(M(I_\kappa)\) be the number of bids in \(I_\kappa\). Let \(\tilde{w} = zw'/24\). Say \(I_\kappa\) is **sparse** if \(E(M(I_\kappa)) < \tilde{w} n^{\alpha/4}\). Let \(X\) be the set of sparse intervals. For \(\kappa \in X\), let \(E_{3\kappa} \equiv \{M(I_\kappa) \leq \frac{3}{z} \tilde{w} n^{\alpha/4}\}\) be the event that there are not “too many” bids in \(I_\kappa\).

For any given \(\tau \in [0, 1]\) consider the process in which at stage one, values and bids are drawn according to the distributional strategy \(\mu\), and at stage two, each bid is randomly and independently replaced by a bid in \(I_\kappa\) with probability \(\tau\). Let \(M_\tau(I_\kappa)\) be the random variable that gives the number of bids in \(I_\kappa\) for this process. Clearly, \(M_\tau(I_\kappa)\) stochastically dominates \(M(I_\kappa)\) for any \(\tau\). Choose \(\tau^*\) such that \(E(M_{\tau^*}(I_\kappa)) = \tilde{w} n^{\alpha/4}\). Then Lemma 2 implies that
\[
\Pr(E_{3\kappa}) \geq \Pr(M_{\tau^*}(I_\kappa) \leq \frac{3}{z} \tilde{w} n^{\alpha/4}) \\
\geq 1 - e^{-\tilde{w} n^{\alpha/4}} \\
\geq 1 - \frac{1}{n^5}
\]
for \(n\) SL.
For \( \kappa \notin X \), let \( E_{3\kappa} \equiv \{ M(I_{\kappa}) \geq 2\tilde{\omega}n^{\alpha/4} \} \) be the event that there are not “too few” bids in \( I_{\kappa} \). Lemma 2 implies that for \( n \) SL,

\[
\Pr(E_{3\kappa}) \geq 1 - e^{-0.3\bar{\omega}n^{\alpha/4}} \geq 1 - \frac{1}{n^3}.
\]

Let \( \mathcal{N}' \equiv \mathcal{N} \cap (\bigcap_{\kappa} E_{3\kappa}) \). Arguing as in the proof of Lemma 3, for \( n \) SL,

(A.22) \[
\Pr(\mathcal{N}') \geq 1 - \frac{3n^{1-\alpha/4}}{n^3} \geq 1 - \frac{1}{n^4}.
\]

**Step 2—Sparse Regions and the Endpoints of Competitive Gaps:** Assemble maximal groups of adjacent sparse intervals into sparse regions. Let \( \{J^\lambda\}_{\lambda \in \Lambda} \) be the set of sparse regions that are longer than \( x/2 \). Note that \( \Lambda \) is possibly empty and that \( |\Lambda| \leq n \). For \( n \) SL, \( 2\tilde{\omega}n^{\alpha/4} > 1 \). So, given \( \mathcal{N}' \), for all \( n \) SL, each non-sparse interval contains at least one bid and so \( cg \) cannot contain a nonsparse interval; \( cg \) can include at most a \( J^\lambda \) and parts of the two non-sparse intervals immediately adjacent. These two intervals, having length at most \( 2/n^{1-\alpha/4} \), become arbitrarily short compared to \( x \gtrsim 1/n^{1-\alpha/2} \) (from (8), \( \alpha_2 > 2\alpha/3 \), and hence \( \alpha_2 > \alpha/4 \)). Hence, given \( \mathcal{N}' \) and for \( n \) SL, a competitive gap of length \( x \) must (a) have intersection of length at least \( x/2 \) with some \( J^\lambda \) and (b) intersect at most one \( J^\lambda \).

Let \( J^\lambda_y, y \in [0, 1] \), be the point a \( y \)th of the way up the interval \( J^\lambda \). We begin by showing that it is very unlikely that \( cg \) ends a long way from the end of a \( J^\lambda \).

**CLAIM 1:** For all \( n \) SL,

\[
\begin{align*}
\Pr\left( \bar{cg} \in \bigcup_{\lambda}[J^\lambda_0, J^\lambda_4/5] \right) &\leq \frac{1}{12n^{2-\alpha_2}x^2}, \\
\Pr\left( cg \in \bigcup_{\lambda}[J^\lambda_1/5, J^\lambda_1] \right) &\leq \frac{1}{12n^{2-\alpha_2}x^2}.
\end{align*}
\]

**PROOF:** Suppose this fails and along a subsequence \( \Pr(H) > 1/(12n^{2-\alpha_2}x^2) \), where \( H \equiv \{ \bar{cg} \in \bigcup_{\lambda}[J^\lambda_0, J^\lambda_4/5] \} \cap \mathcal{N}' \) for some \( \lambda \in \Lambda \). Let \( y \equiv J^\lambda_y = \bar{cg} \). Since \( \bar{cg} \in [J^\lambda_0, J^\lambda_4/5] \), \( y/2 \geq x/20 \gtrsim 1/(20n^{1-\alpha/2}) \) by (A.21). Recall from (8) that \( \alpha_2 > (2/3)\alpha \); thus, for \( n \) SL, \( y/2 > 1/n^{1-\alpha/3} \). By Definition 2, therefore, the number of values in \( [J^\lambda_1 - y/2, J^\lambda_1] \) is at least \( w'y/n/2 \).

On the other hand, by Step 1, given \( \mathcal{N}' \), each \( I_{\kappa} \subseteq [J^\lambda_y - y, J^\lambda_y] \) includes at most \( \frac{3}{2} \tilde{\omega}n^{\alpha/4} = \frac{3}{2}(zw'/24)n^{\alpha/4} = w'n^{\alpha/4}/8 \) bids. For \( n \) SL, this implies that the number of bids in \( [J^\lambda_1 - y, J^\lambda_1] \) is at most \( w'y/n/4 \) (by the same argument as in the proof of Lemma 3). Thus, given \( \{ \bar{cg} \in [J^\lambda_0, J^\lambda_4/5] \} \cap \mathcal{N}' \), there are at least
\[ w'y_n/2 - w'y_n/4 = w'y_n/4 \] players with value in \([J_1^\lambda - y/2, J_1^\lambda]\) but bid below \(c_g = J_1^\lambda - y\). Then
\[
Y_B\left(\frac{y}{2}\right) \geq \frac{w'y_n y}{4}. \]

Since \(c_g \in [J_0^\lambda, J_4^\lambda]\), \(y \geq x/10\). So, for any given \(\lambda\), whenever \(\{c_g \in [J_0^\lambda, J_4^\lambda]\} \cap N'\),
\[
Y_B\left(\frac{x}{20}\right) \geq Y_B\left(\frac{y}{2}\right) \geq \frac{w'n x^2}{800}. \]

Now, for \(n\) SL,
\[
\Pr(H \cap N') \geq \Pr(H) - \frac{1}{n^4} \geq \frac{1}{2} \Pr(H), \]

since for \(n\) SL, \(\frac{1}{n^4} < \frac{1}{2}(1/(12n^{2-a_2}x^2)) \leq \frac{1}{2} \Pr(H)\). Thus,
\[
E\left(Y_B\left(\frac{x}{20}\right)\right) \geq \frac{w'n x^2}{1,600} \Pr(H). \]

By (A.21), for \(n\) SL, \(\frac{1}{n^4} > \frac{1}{n}\) and, hence, \(Y_B\left(\frac{x}{20}\right) < Y_B\left(\frac{1}{n}\right)\). However, \(E(Y_B(1/n)) \leq 1/n^{1-a_3}\) by Lemma 7. Thus
\[
\frac{w'n x^2}{1,600} \Pr(H) \leq \frac{1}{n^{1-a_3}}. \]

Rearranging yields
\[
\Pr(H) \leq \frac{1,600}{n^{2-a_3}x^2w'} \leq \frac{1}{12n^{2-a_2}x^2} \]

for \(n\) SL.\(^{23}\) This contradicts our initial assertion. Repeat for sellers in the lower fifth to get the second claim. \(Q.E.D.\)

**Step 4—Relative Probabilities of Competitive and Supporting Gaps:** Let \(cg_\lambda \equiv \{cg \supseteq [J_1^\lambda, J_4^\lambda]\}\) and let \(c_\lambda \equiv \Pr(cg_\lambda).\) Let \(lg_\lambda \equiv \{lg \supseteq [J_1^\lambda, J_2^\lambda]\}\) and \(l_\lambda \equiv \Pr(lg_\lambda).\) Finally, let \(ug_\lambda \equiv \{ug \supseteq [J_3^\lambda, J_4^\lambda]\}\) and \(u_\lambda \equiv \Pr(ug_\lambda).\) We next show that for some \(\lambda\), \(c_\lambda\) is both nontrivial and much larger than either \(l_\lambda\) or \(u_\lambda\).

\(^{23}\)We remark again that we have chosen transparency over any attempt to keep the various constants in these arguments small.
CLAIM 2: For $n$ SL, there exists $\lambda$ such that

\[ c_\lambda > \frac{1}{n^4} \]  

and such that

\[ \frac{l_\lambda + u_\lambda}{c_\lambda} \leq \frac{1}{n^{(a_2-a_3)/2}}. \]  

PROOF: Since $\Pr(|cg| \geq x) > n^{a_2-2}x^{-2}$ and $\Pr(\mathcal{N'}) \geq 1 - 1/n^4$, for $n$ SL, $\Pr(|cg| \geq x) \cap \mathcal{N'} \geq \frac{5}{6}n^{a_2-2}x^{-2}$. By Claim 1, the probability of a competitive gap in $J^\lambda$ not including the middle $3/5$ is also less than $\frac{1}{6}n^{a_2-2}x^{-2}$ for $n$ SL. Therefore, $\sum_{\lambda \in \Lambda} c_\lambda \geq \frac{5}{6}n^{a_2-2}x^{-2}$. Let $\Lambda'$ denote the subset of regions with $c_\lambda > 1/n^4$. There are at most $n$ regions. Thus,

\[ \sum_{\lambda \in \Lambda \setminus \Lambda'} c_\lambda \leq \frac{n}{n^4} \leq \frac{1}{6}n^{a_2-2}x^{-2} \]

for $n$ SL and so

\[ \sum_{\lambda \in \Lambda'} c_\lambda \geq \frac{1}{2}n^{a_2-2}x^{-2}. \]

From Lemma 6, for $n$ SL,

\[ \sum_{\lambda \in \Lambda'} u_\lambda - \frac{1}{n^3} \leq \Pr(|ug| > \frac{x}{5}) \leq 25n^{a_3-2}x^{-2}. \]

The first inequality holds because under $\mathcal{N'}$ every nonsparse interval contains at least one bid, and so $ug_\lambda$ and $ug_{\lambda'}$ are disjoint events. When $\mathcal{N'}$ does not hold, the overcounting is at most $|\Lambda'| \leq n$. Whereas $\Pr(\mathcal{N'}^c) \leq 1/n^4$ for $n$ SL, the result follows. The second inequality applies Lemma 7. Then

\[ \sum_{\lambda \in \Lambda'} l_\lambda + u_\lambda \leq 50n^{a_3-2}x^{-2} + \frac{2}{n^3}. \]

Thus

\[ \frac{\sum_{\lambda \in \Lambda'} l_\lambda + u_\lambda}{\sum_{\lambda \in \Lambda'} c_\lambda} \leq \frac{50n^{a_3-2}x^{-2} + \frac{2}{n^3}}{\frac{1}{2}n^{a_2-2}x^{-2}} \leq \frac{1}{n^{(a_2-a_3)/2}}. \]

for $n$ SL. Since this is true on average, it must be true for at least one $\lambda \in \Lambda'$.

\[ Q.E.D. \]
In what follows, we refer to a \( \lambda \) for which Claim 2 holds. Let \( \tilde{c} \equiv \Pr(c g \supseteq [J_{\lambda \cdot 2/5}, J_{\lambda \cdot 3/5}]) \) be the probability of a competitive gap including the middle fifth of \( J^{\lambda} \). We will show that \( \tilde{c} \) is close to 1. The idea is that the only way to have \( c_{\lambda} \) be large relative to \( u = u_{\lambda} \) and \( l = l_{\lambda} \) will be for players to behave essentially deterministically.

Let \( U_i \equiv \{ b_i \geq J_{\lambda \cdot 4/5} \} \) be the event that \( i \) bids up and let \( \tilde{U}_i \equiv \{ b_i \geq J_{\lambda \cdot 2/5} \} \) be the event that \( i \) bids weakly up. Symmetrically, let \( D_i \equiv \{ b_i \leq J_{\lambda \cdot 1/5} \} \) and \( \tilde{D}_i \equiv \{ b_i \leq J_{\lambda \cdot 3/5} \} \) be the events that \( i \) bids down and weakly down. Let \( p_i \equiv \Pr(U_i) \), \( \tilde{p}_i \equiv \Pr(\tilde{U}_i) \), \( q_i \equiv \Pr(D_i) \), and \( \tilde{q}_i \equiv \Pr(\tilde{D}_i) \). Order the players so that \( \tilde{q}_1 \leq \tilde{q}_2 \leq \cdots \leq \tilde{q}_{2n} \).

**Step 5—A Preliminary Inequality:** Define

\[
A_{\cdots \cdots \cdots i-1} = \bigcap_{j' > i, j' \neq j} D_{j'}
\]

as the event that all players after \( i \) not including \( j \) bid down. Then, for any event \( F \) involving \( 1, 2, \ldots, i \),

\[
\Pr(A_{\cdots \cdots \cdots i-1} \mid F) = \prod_{j' > i, j' \neq j} \Pr\left( D_{j'} \mid \bigcap_{j'' \in \{i+1, \ldots, i'-1\}\setminus j} D_{j''}, F \right)
\]

\[
= \exp\left[ \sum_{j' > i, j' \neq j} \log \Pr\left( D_{j'} \mid \bigcap_{j'' \in \{i+1, \ldots, i'-1\}\setminus j} D_{j''}, F \right) \right]
\]

\[
\leq \exp\left[ \sum_{j' > i, j' \neq j} \Pr\left( D_{j'} \mid \bigcap_{j'' \in \{i+1, \ldots, i'-1\}\setminus j} D_{j''}, F \right) - 1 \right]
\]

\[
\leq \exp\left[ - \sum_{j' > i, j' \neq j} \Pr\left( D_{j'}^c \mid \bigcap_{j'' \in \{i+1, \ldots, i'-1\}\setminus j} D_{j''}^c, F \right) \right]
\]

\[
= \exp\left( -z \sum_{j' > i, j' \neq j} 1 - q_{j'} \right) \quad \text{(by } z\text{-independence)}
\]

\[
\leq \exp\left( -z \sum_{j' > i, j' \neq j} p_{j'} \right) \quad \text{(since } 1 - q_{j'} \geq p_{j'})
\]

\[
\leq \exp\left( -z \left( -1 + \sum_{j' > i} p_{j'} \right) \right).
\]
Step 6—Two Bounds: Recall that $c_{g,\lambda} \equiv \{c_g \supseteq [J_{i,j}^\lambda, J_{i,j}^\lambda]\}$. Let $c_{g,j}^{ij} = \Pr(c_{g,j}^{ij})$. Let $F^i$ be the event that $U_{i'}$ holds for $i' = i$ and for $n - 2$ other $j' \in \{1, \ldots, i\}$, while $D_{j'}$ holds for all other $j' \in \{1, \ldots, i\}$. Then
\[
c_{g,j}^{ij} = \Pr(F^i \cap A_{i,j}^j \cap U_j) = \Pr(U_j | F^i \cap A_{i,j}^j) \Pr(F^i \cap A_{i,j}^j) = \frac{\Pr(\tilde{D}_j | F^i \cap A_{i,j}^j)}{\Pr(\tilde{D}_j | F^i \cap A_{i,j}^j)} \Pr(U_j | F^i \cap A_{i,j}^j) \Pr(F^i \cap A_{i,j}^j) \leq \frac{p_j}{z^2 \tilde{q}_j} \Pr(\tilde{D}_j \cap F^i \cap A_{i,j}^j)
\]
Recall that $u_{g,\lambda}$ is the event $\{u_g \supseteq [J_{3/5}^\lambda, J_{4/5}^\lambda]\}$. Let $u_{g,j}^i$ be the event $u_{g,\lambda}$ where $i$ is the last player to bid up and let $u_{\lambda}^i = \Pr(u_{g,j}^i)$. When $\tilde{D}_j \cap F^i \cap A_{i,j}^j$ holds, $i$ is the last player to bid up and in total $n - 1$ players bid up while the rest bid weakly down. Thus $\{\tilde{D}_j \cap F^i \cap A_{i,j}^j\} \subseteq u_{g,j}^i$ and so $\Pr(\tilde{D}_j \cap F^i \cap A_{i,j}^j) \leq u_{\lambda}^i$. The previous equation thus implies
\[(A.26) \quad c_{g,j}^{ij} \leq \frac{p_j}{z^2 \tilde{q}_j} u_{\lambda}^i.\]

Another bound on $c_{g,j}^{ij}$ comes from (A.25):
\[(A.27) \quad c_{g,j}^{ij} \leq \Pr(A_{i,j}^j) \leq \exp\left(-z \left(-1 + \sum_{j' > i} p_{j'}\right)\right).\]

Step 7—Up and Down Players: We next show that for all $n$ SL, $\tilde{q}_i = \Pr(\tilde{D}_i) \leq \frac{1}{2}$ for $i \leq n$ (these are the up players) and $\tilde{p}_i = \Pr(\tilde{U}_i) \leq \frac{1}{2}$ for $i > n$ (these are the down players).

Consider the first claim. Suppose that $\tilde{q}_i > 0$ (if $\tilde{q}_n = 0$, the result is immediate). Let $c_{g,i}^i \equiv \sum_{j > i} c_{g,j}^{ij}$ be the probability that $c_{g,\lambda}$ occurs, where $i$ is the second last up player. Let $i^* \leq 2n$ be the last index with the property that $\sum_{j' > i^*} p_{j'} > n^{(z^2 - 2\alpha)}/\lambda$. Then
\[
\sum_{i \leq i^*} c_{g,i}^i = \sum_{i \leq i^*} c_{g,j}^{ij} \leq n \exp\left(-z \left(-1 + \sum_{j' > i^*} p_{j'}\right)\right) \leq n \exp\left(-z \left(-1 + \sum_{j' > i^*} p_{j'}\right)\right) \text{ (using (A.27))}
\]
\[ \leq n^2 \exp \left( -z \left( -1 + n^{(\alpha_2 - \alpha_3)/4} \right) \right) \]

\[ \leq \frac{1}{2n^4} \quad \text{(for } n \text{ SL)} \]

\[ \leq \frac{1}{2} c_{\lambda} \quad \text{(using (A.23))} \]

So, as \( c_{\lambda} = \sum_{i \geq n-1} c_{\lambda}^i \), for \( n \) SL,

\[ \frac{1}{2} c_{\lambda} \leq \sum_{i \geq i^*, i \geq n-1} c_{\lambda}^i \]

\[ \leq \sum_{i \geq i^*, i \geq n-1} \sum_{j > i} \frac{p_j}{z^2 \tilde{q}_j} u_{\lambda}^i \quad \text{(using (A.26))} \]

\[ \leq \frac{1}{z^2 \tilde{q}_n} \sum_{i \geq i^*, i \geq n-1} \sum_{j > i} p_j u_{\lambda}^i \quad \text{(since } \tilde{q}_i \text{ is increasing)} \]

\[ \leq \frac{n^{(\alpha_2 - \alpha_3)/4}}{z^2 \tilde{q}_n} \sum_{i \geq i^*, i \geq n-1} u_{\lambda}^i \quad \text{(by choice of } i^*) \]

\[ \leq \frac{n^{(\alpha_2 - \alpha_3)/4}}{z^2 \tilde{q}_n} u_{\lambda} \]

\[ \leq \frac{n^{(\alpha_2 - \alpha_3)/4}}{z^2 \tilde{q}_n} \frac{1}{n^{(\alpha_2 - \alpha_3)/2} c_{\lambda}} \quad \text{(by (A.24))} \]

\[ = \frac{1}{z^2 \tilde{q}_n} \frac{1}{n^{(\alpha_2 - \alpha_3)/4} c_{\lambda}}. \]

Comparing the first and last expressions, \( \tilde{q}_n \to 0 \) and so, in particular, \( q_i \leq 1/4 \) all \( i \leq n \) for all \( n \) SL.

If the players are ordered so that \( \tilde{p}_i \) increases, this argument can be repeated considering events in which \( n - 1 \) of the first \( i \) players bid down and the others bid up. Thus there are \( n \) players for whom \( \tilde{p}_i \to 0 \). Whereas \( \tilde{p}_i + \tilde{q}_i \geq 1 \), these players are disjoint from players \( 1, \ldots, n \), and so must be the players \( \{n + 1, \ldots, 2n\} \).

Let \( R \equiv \bigcap_{i \leq n} U_i \bigcap_{i > n} D_i = c_{\lambda}^{n-1,n} \) be the event that all the players bid according to their type (and a competitive gap occurs).

**Step 8—A Lower Bound for \( \Pr(R) \):** We already know that \( \sum_{i \leq i^*, j > i} c_{\lambda}^{ij} \leq \frac{1}{2} c_{\lambda} \) for \( n \) SL. We will show that for \( n \) SL, \( \sum_{i > i^*, j > n} c_{\lambda}^{ij} \leq \frac{1}{4} c_{\lambda} \). Since \( R \) is the only event left that involves \( c_{\lambda} \), it would then follow that \( \Pr(R) \geq c_{\lambda}/4 \). So, as in
Step 7, note that

\[ \sum_{i \neq i^*, j \geq n+1} c^{ij}_\lambda \leq \sum_{i \neq i^*, j \geq n+1} \frac{p_j}{z^2 \hat{q}_j} u^i_\lambda \]

\[ \leq \frac{1}{z^2 \hat{q}_{n+1}} \sum_{i \neq i^*} u^i_\lambda \sum_{j \neq i} p_j \quad \text{(note the } n + 1) \]

\[ \leq \frac{n^{(a_2 - a_3)/4}}{2z^2} \sum_{i \neq i^*} u^i_\lambda \quad \text{(since } \hat{q}_{n+1} \geq 1 - \tilde{p}_{n+1} \geq 3/4) \]

\[ \leq \frac{n^{(a_2 - a_3)/4}}{2z^2} u_\lambda \]

\[ \leq \frac{n^{(a_2 - a_3)/4}}{2z^2} \frac{1}{n^{(a_2 - a_3)/2}} c_\lambda \]

\[ \leq \frac{1}{2z^2} \frac{1}{n^{(a_2 - a_3)/4}} c_\lambda \]

\[ \leq \frac{1}{4} c_\lambda \quad \text{(for } n \text{ SL).} \]

Thus, \( \Pr(R) \geq c_\lambda / 4. \)

**Step 9—A Persistent Competitive Gap:** Let \( \tilde{R} \supseteq R \) be the event that all the players get it nearly right—the first \( n \) players are not bidding below \( J^\lambda_{3/5} \) and the others are not bidding above \( J^\lambda_{2/5} \). For \( i > n \), define \( R_{-i} \) to be the event that all players except \( i \) play according to type. If \( R_{-i} \) occurs and player \( i \) bids weakly up, then \( \tilde{l}_g \supseteq [J^\lambda_{1/5}, J^\lambda_{2/5}] \). Thus,

\[ u_\lambda \geq \sum_{i > n} \Pr(R_{-i}) \Pr(\tilde{U}_i | R_{-i}) \]

\[ \geq z \Pr(R) \sum_{i > n} \tilde{p}_i \quad \text{(by } z\text{-independence and since } R \subseteq R_{-i}) \]

\[ \geq \frac{zc_\lambda}{4} \sum_{i > n} \tilde{p}_i \quad \text{(by Step 8).} \]

Since \( u_\lambda / c_\lambda \to 0 \),

\[ \sum_{i > n} \tilde{p}_i \to 0. \]
Arguing symmetrically,
\[ \sum_{i \leq n} \tilde{q}_i \to 0. \]

Thus
\[ \Pr(\tilde{R}) \geq 1 - \sum_{i \leq n} \tilde{q}_i - \sum_{i > n} \tilde{p}_i \to 1. \]

**Step 10—A Contradiction:** When \( \tilde{R} \cap \mathcal{N}' \) occurs, \( [J_{2/5}^\lambda, J_{3/5}^\lambda] \subseteq cg \). Since the probability of trade is bounded away from 0 and since \( \Pr(\tilde{R}) \to 1 \), there is at least one buyer in \( \{1, \ldots, n\} \) and at least one seller in \( \{n + 1, \ldots, 2n\} \).

Let \( p^* \) be the expected price conditional on \( \tilde{R} \). Either \( p^* \leq J_{1/2}^\lambda \) or \( p^* \geq J_{1/2}^\lambda \).

Without loss of generality, assume \( p^* \geq J_{1/2}^\lambda \). Let \( x^\lambda \) be the length of \( J^\lambda \). By construction, \( x^\lambda \geq \frac{1}{2}x \geq 1/(2n^{1-\alpha_2/2}) \).

Assume first that \( J_{1/2}^\lambda \geq 1 - 3x^\lambda \). Consider any buyer in \( \{1, \ldots, n\} \). A bid of \( J_{2/5}^\lambda \) wins whenever \( \tilde{R} \) occurs, and forces the price to at most \( J_{2/5}^\lambda \). So, conditional on \( \tilde{R} \), the buyer’s expected gain from lowering the price is at least \( J_{1/2}^\lambda - J_{2/5}^\lambda \geq x^\lambda/6 \). On the other hand, when \( \tilde{R} \) does not occur, he may go from being a winner to a loser. For this to happen, it must be that \( cg \geq J_{2/5}^\lambda \). However, then \( i \)'s lost profit is at most \( 1 - cg \leq 4x^\lambda \). Since \( \Pr(\tilde{R}) \to 1 \),
\[ \Pr(\tilde{R}) \frac{x^\lambda}{6} - (1 - \Pr(\tilde{R}))4x^\lambda \]
is eventually positive and we have a contradiction.

Assume \( J_{1/2}^\lambda < 1 - 3x^\lambda \). Given \( \mathcal{N}' \), the number of buyers with value in \( (J_{1/2}^\lambda + 2x^\lambda, J_{1/2}^\lambda + 3x^\lambda) \) is at least \( w'n x^\lambda \), but by Lemma 7 for \( n \) SL,
\[ E(\#U(x^\lambda)) \leq \frac{1}{n^{1-\alpha_4} x^\lambda}. \]

Since \( x^\lambda \geq 1/(2n^{1-\alpha_2/2}) \),
\[ \frac{w'n x^\lambda}{1/(n^{1-\alpha_4} x^\lambda)} \geq w'n^{\alpha_2 - \alpha_4}. \]

It follows that for \( n \) SL, at least half the buyers with a value in \( (J_{1/2}^\lambda + 2x^\lambda, J_{1/2}^\lambda + 3x^\lambda) \) trade conditional on \( \tilde{R} \cap \mathcal{N}' \) (and so bid above \( J_{3/5}^\lambda \)). Consider the deviation that any buyer with a value in \( (J_{1/2}^\lambda + 2x^\lambda, J_{1/2}^\lambda + 3x^\lambda) \) and a bid above \( J^\lambda \) bids \( J_{2/5}^\lambda \) instead. Given \( \tilde{R} \cap \mathcal{N}' \), this gains the buyer at least \( x^\lambda/6 \). Given \( \mathcal{N}' \), the number of players in \( (J_{1/2}^\lambda + 2x^\lambda, J_{1/2}^\lambda + 3x^\lambda) \) is at least \( w'n x^\lambda \) and at most \( W'n x^\lambda \).
So, given \( R \cap \mathcal{N}' \), the expected sum of gains is at least \((w'nx^\lambda/2)(x^\lambda/6)\). The loss from such a buyer going from being a winner to a loser is again at most \(4x^\lambda\).

Given \( \mathcal{N}' \setminus \tilde{R} \), there are at most \(W'nx^\lambda\) such buyers. In \( \mathcal{N}'' \), the worst case is that all \(n\) buyers are in \((J_1^\lambda + 2x^\lambda, J_1^\lambda + 3x^\lambda)\), so the expected sum of losses is at most

\[
\Pr(\mathcal{N}' \setminus \tilde{R})W'nx^\lambda4x^\lambda + \Pr(\mathcal{N}'')n4x^\lambda
\]

and thus, since the deviation cannot be profitable,

\[
\Pr(\tilde{R}) \frac{w'nx^\lambda}{2} \frac{x^\lambda}{6} \leq (1 - \Pr(\tilde{R}))W'nx^\lambda(x^\lambda)^2 + \frac{1}{n^4} 4x^\lambda
\]

Dividing both sides by \(n(x^\lambda)^2\) yields

\[
\Pr(\tilde{R}) \frac{w'}{12} \leq 4W'(1 - \Pr(\tilde{R})) + \frac{4}{n^2x^\lambda}
\]

\[
\leq 4W'(1 - \Pr(\tilde{R})) + \frac{8}{n^2x^\lambda} \quad \text{(since } x^\lambda \geq (1/2)x) \]

\[
\leq 4W'(1 - \Pr(\tilde{R})) + \frac{8}{n^{2+\alpha/2}}
\]

\[
= 4W'(1 - \Pr(\tilde{R})) + \frac{8}{n^{1+\alpha/2}}
\]

Since \(\Pr(\tilde{R}) \to 1\), the left-hand side goes to \(w'/12\), while the right-hand side goes to 0—a contradiction.

**Q.E.D.**

**PROOF OF THEOREM 2:** Let \(x\) be the random variable \(\overline{u_g}m - cg\). In an \(m\)-unit demand/supply setting, this is the maximum impact of raising a buyer’s bid vector on price. Let \(p\) be the price. We will show that in expectation buyers achieve within \(1/2n^{1-\alpha}\) of the consumer surplus if they can price take at \(p\).

A symmetric argument applies to sellers. However, the sum of consumer and producer surplus at an arbitrary \(p\) is at least as large as the surplus at the Walrasian price. So this both establishes that the market achieves within \(1/n^{1-\alpha}\) of the efficient surplus and that price must be asymptotically Walrasian, else the market achieves more than the feasible surplus—a contradiction.\(^{24}\) Finally, \(\Pr(\tilde{R}) \to 1\), the left-hand side goes to \(w'/12\), while the right-hand side goes to 0—a contradiction. **Q.E.D.**

\(^{24}\)Formally, the difference between the realized price and the competitive price must converge to zero in probability.
from NAG and NAA, expected feasible surplus grows like $n$ and the result follows.

Consider, for buyers, the truth-telling deviation $d_i(b_i, v_i) = v_i$, remembering that $v_i$ and $b_i$ are now vectors in $[0, 1]^m$. Let $W$ be the set of $ih$, $i \in N_h$, that are allocated an object. Let $Y_{ih} = 0$ if $i$ wins an object $h$ and let $Y_{ih} = \max[v_{ih} - p, 0]$ otherwise. So $Y_{ih}$ gives the loss in consumer surplus compared to price taking at $p$ from $i$ not winning object $h$.

In any given realization, think about moving from $b_i$ to $v_i$ one bid at a time, starting from $b_{i1}$. Let $\hat{C}_{ih}$ be the cost to $i$ from raising bid $h$ in terms of raising the price paid on units already won and let $\hat{B}_{ih}$ be the profit to $i$ of winning an extra unit.

If $v_{ih} < c_g$, then raising $b_{ih}$ to $v_{ih}$ is irrelevant to both $p$ and the allocation. Hence, $\hat{B}_{ih} - \hat{C}_{ih} = 0$ and, since $v_{ih} < p$, $Y_{ih} = 0$.

If $v_{ih} \in [c_g, \bar{u}g + 2mx]$, then raising $b_{ih}$ to $v_{ih}$ may raise the price on units already won by as much as $x$. So $\hat{B}_{ih} - \hat{C}_{ih} \geq -(m - 1)x$, and since $v_{ih} \in [c_g, \bar{u}g + 2mx]$ and $p \geq c_g$, $Y_{ih} \leq (2m + 1)x$. In any normal realization, the number of such $ih$ is at most $Knx$ for some $K < \infty$. In a nonnormal realization, there are at most $nm$ values in this range. Hence, the expected number of such values is at most

$$\left(1 - \frac{1}{n^4}\right)Knx + \frac{1}{n^4}nm \leq Knx + \frac{m}{n^3}.$$  

Consider $ih \in W$ such that $v_{ih} > \bar{u}g + 2mx$. In any realization, at most one player who is winning an object is also in a position to affect the price by changing the associated bid, and the impact of that bid on price is at most $x$. Hence,

$$\sum_{\{ih \in W | v_{ih} > \bar{u}g + 2mx\}} \hat{C}_{ih} \leq x$$

and

$$\sum_{\{ih \in W | v_{ih} > \bar{u}g + 2mx\}} Y_{ih} = 0.$$  

Finally, consider $ih \notin W$ such that $v_{ih} > \bar{u}g + 2mx$. Then, by deviating to $b_{ih} = v_{ih}$, $i$ raises the price on at most $m - 1$ previous units by at most $x$. However, $i$ also wins an extra object at price at most $\bar{u}g$, so

$$\hat{B}_{ih} - \hat{C}_{ih} \geq v_{ih} - \bar{u}g - (m - 1)x \geq \frac{v_{ih} - p}{2} = \frac{Y_{ih}}{2}.$$
Since we are in equilibrium,

\[(A.28) \quad 0 \geq E\left( \sum_{ih} \hat{B}_{ih} - \hat{C}_{ih} \right) \]

\[= E\left( E\left( \sum_{(ih) \notin W, \vih > \ug + 2mx} \hat{B}_{ih} - \hat{C}_{ih} \mid x \right) \right) + E\left( E\left( \sum_{(ih) \in W, \vih > \ug + 2mx} \hat{B}_{ih} - \hat{C}_{ih} \mid x \right) \right) + E\left( E\left( \sum_{(ih) \notin W, \vih \in \{ \vih \mid \vih > \ug + 2mx \}} \hat{B}_{ih} - \hat{C}_{ih} \mid x \right) \right) \]

\[\geq E\left( E\left( \sum_{(ih) \notin W, \vih > \ug + 2mx} \frac{Y_{ih}}{2} \mid x \right) \right) - E(x) - E\left( E\left( \sum_{(ih) \in W, \vih \in \{ \vih \mid \vih > \ug + 2mx \}} (m - 1)x \mid x \right) \right) \]

\[\geq E\left( E\left( \sum_{(ih) \notin W, \vih > \ug + 2mx} \frac{Y_{ih}}{2} \mid x \right) \right) - E(x) - E\left( \left( Knx + \frac{m}{n^3} \right) (m - 1)x \right) \]

\[\geq E\left( E\left( \sum_{(ih) \notin W, \vih > \ug + 2mx} \frac{Y_{ih}}{2} \mid x \right) \right) - 2E(x) - K''nE(x^2) \quad \text{(for } n \text{ SL).} \]

Let $H$ be the cumulative for $x$. By Lemmas 6 and 8, $H(x) \leq 2/(n^{2-a_2}x^2)$ for all $x$. Hence,

\[nE(x^2) = \int_0^1 nx^2 dH(x) \]

\[= \int_0^1 n2x[1 - H(x)] dx \]

\[\leq \int_0^{1/n} 2nx dx + \int_{1/n}^1 2nx \frac{2}{n^{2-a_2}x^2} dx \]
\[
\begin{align*}
&= nx^{2\gamma} \int_{0}^{1/n} \frac{1}{x} \, dx \\
&\leq \frac{1 + 4 \log n}{n^{1-a_2}} \\
&\leq \frac{1}{n^{1-a_1}} \quad \text{(for } n \text{ SL)}
\end{align*}
\]

and \(E(x) \leq 1/n^{1-a_1}\) as well (a simple integration by parts). So (A.28) yields

\[
E\left( E\left( \sum_{\substack{i \mid v_{ih} > \bar{v} + 2mx}} Y_{ih} \mid x \right) \right) \leq 2\left( \frac{1}{n^{1-a_1}} + K^{''} \frac{1}{n^{1-a_1}} \right)
\]

\[
= K^{'''} \frac{1}{n^{1-a_1}}.
\]

However, then

\[
E\left( \sum_{i} Y_{ih} \right) = E\left( E\left( \sum_{\substack{i \mid v_{ih} > \bar{v} + 2mx}} Y_{ih} \mid x \right) \right)
\]

\[
+ E\left( E\left( \sum_{v_{ih} \in [c_{0}, \bar{v} + 2mx]} Y_{ih} \mid x \right) \right)
\]

\[
\leq K^{'''} \frac{1}{n^{1-a_1}} + E\left( Knx + \frac{m}{n^3} \right) (2m + 1)x
\]

\[
\leq K^{'''} \frac{1}{n^{1-a_1}} + \frac{2m + 1}{n^3} mE(x) + K^{''''} E(nx^2)
\]

\[
\leq \frac{1}{2n^{1-a}} \quad \text{for } n \text{ SL.}
\]

Arguing analogously for sellers, \(E(\sum_{i} Y_{ih}) \leq 1/(2n^{1-a})\). Hence, the expected sum of consumer and producer surplus is within \(1/n^{1-a}\) of that achieved by the Walrasian outcome, and we are done. \(Q.E.D.\)

REFERENCES


