On the analysis of asymmetric first price auctions

Vlad Mares\textsuperscript{a}, Jeroen M. Swinkels\textsuperscript{b,\!*}

\textsuperscript{a} INSEAD, France
\textsuperscript{b} Kellogg School of Management, Northwestern University, Evanston, IL, United States

Received 6 May 2011; final version received 23 October 2012; accepted 12 February 2013
Available online 28 March 2014

Abstract
We provide new tools for studying asymmetric first price auctions, connecting their equilibria to the $\rho$-concavity of the underlying type distributions, and showing how one can use surplus expressions for symmetric auctions to bound equilibrium behavior in asymmetric auctions. We apply these tools to studying procurement auctions in which, as is common in practice, one seller is given an advantage, reflecting for example better reliability or quality. We show conditions under which for any given first price handicap auction, there exists a superior second price auction with bonuses.

© 2014 Elsevier Inc. All rights reserved.

\textit{JEL classification:} D44; D47; D82

\textit{Keywords:} Asymmetric auctions; Request for proposal; Differentiation; Mechanism design; First price auctions; Second price auctions; Procurement; Rho-concavity

1. Introduction

In many auction settings, players come from different type distributions. And, even when they come from similar distributions, the auctioneer may have a preference over with whom to transact. For example, a buyer may view two sellers as having different quality or reliability of delivery.

\textsuperscript{*} Corresponding author.

http://dx.doi.org/10.1016/j.jet.2014.03.010
0022-0531/© 2014 Elsevier Inc. All rights reserved.
Real-world auction practice reflects these asymmetries. Ariba.com, a leading Internet-based provider of procurement auctions, allows its customers to use fixed handicaps and bonuses in combination with either first or second price (open) rules. The standard sealed bid request for proposal is equivalent to a first price auction with a handicap equal to the amount by which the auctioneer prefers his favored bidder. Similarly, an open request for proposal is equivalent to a second price bonus auction again with the bonus equal to the amount by which the auctioneer prefers his favored bidder. When quality is exogenous, auctions with quasi-linear scoring rules also reduce to simple handicaps and bonuses. The World Bank [50,51] uses these auctions themselves and requires them of its loan recipients. Such auctions have subsequently become standard for a majority of NGOs and for a host of other international agencies. They are also widely used in corporate procurement by firms such as Boeing and by Ariba competitor Perfect Commerce.

A better understanding of asymmetric auctions is thus of the first importance. How, for example, should one think about auction format choice in such a setting? The difficulty is that once asymmetry is introduced, first price auctions in particular become notoriously difficult to analyze. This paper provides new tools for understanding such auctions, with a focus on the two-bidder setting.

At the heart of our results is a strong connection between the equilibria of asymmetric first price auctions, their induced surplus functions and a measure of the concavity of the distributions over types. This measure, local $\rho$-concavity, has appeared in a number of guises in multiple fields of economic analysis, although it is typically not identified as such. It plays a central role in both Anderson and Renault [1] and Weyl and Fabinger [52], who study oligopoly settings. In fact, as the later authors point out, an expression equal to one minus the local concavity appears prominently in Cournot’s [12] study of the oligopoly problem. For a utility function, the local concavity of the marginal utility of income is in fact the ratio of prudence to absolute risk aversion. Further, as we show below, there is a tight connection between local concavity and relative risk aversion. In particular, if the inverse of a function shows increasing (decreasing) relative risk aversion, then the function will show increasing (decreasing) local concavity. The concept of $\rho$-concavity is also central to Caplin and Nalebuff’s results in social choice [8,9].

We begin by characterizing the unique equilibrium for our setting, and showing that each player’s bidding function has slope equal to 1 minus the local $\rho$-concavity of his interim expected surplus function. For the symmetric case, this reduces to 1 minus the $\rho$-concavity of the distribution of bidder types. Based on this, we use $\rho$-concavity to explore how increases in surplus are divided between the seller and the buyer. We give a tight answer to the question of when a “better” distribution over types leads to more aggressive bidding for any given type. We also use the properties of $\rho$-concavity to generate new comparative statics on equilibria in symmetric settings.

We then turn to the specific setting of an auctioneer who prefers to transact with one bidder over the other and needs to choose an auction format. Two simple mechanisms in this setting are a first price and a second price auction each with a handicap or bonus. As illustrated by the introductory examples, both types of auction are very common in practice, and so a comparison of their merits is of substantial importance.

---

1 The local $\rho$-concavity of a positive function $g$ at $x$ is the power to which one needs to raise $g$ to make it locally linear at $x$.
2 Wilson [53] refers to this quantity as cautiousness.
3 We discuss examples further in the literature review and in Section 4.
First price handicap auctions in this setting have an odd feature. They create an allocation with lots of distortion relative to the symmetric allocation when types are unfavorable (when buyers have high cost, or sellers have low value), but very little when types are favorable (low costs or high values). This is always true at the extremes of the support of types, and we use the tools of $\rho$-concavity to exhibit conditions under which the distortion is in fact monotonically increasing as types become less favorable. The same tools allow us to derive conditions under which the optimal mechanism (Myerson [40], Riley and Samuelson [46]) specifies a distortion that is decreasing as types become less favorable, so that the first price mechanism is providing the most distortion precisely where an optimal mechanism would provide the least. A second price mechanism, on the other hand, creates a uniform distortion away from the symmetric cost case. Combining these results, we show that for any given first price mechanism there is a well-designed second price mechanism that outperforms it. Of course, this means that the best second price mechanism outperforms any first price mechanism.

Choosing a sensible handicap in a first price setting is hard in practice, not least because simply performing the equilibrium calculations for any given handicap is a daunting task. And, given the difficulty of these calculations, the auctioneer may be skeptical that players will reliably find an equilibrium for the handicap chosen, something which may or may not work to the auctioneer’s advantage. Equilibrium calculations for a second price auction, on the other hand, are trivial for both the auctioneer and the players, with potentially beneficial effects on participation.

We thus view this paper as contributing to the Wilson agenda of finding and justifying simple mechanisms, since an implication of our results is that in many settings, the auctioneer can ignore first price mechanisms in favor of the much simpler task of searching for a good second price mechanism. In related work (Mares and Swinkels [30]), we show how the tools of $\rho$-concavity make this search practical, allowing the auctioneer to translate a surprisingly small amount of information about the underlying type distributions into a recommended handicap that performs well even compared to the optimal mechanism. While our model surely does not capture all procurement settings, it seems a fair approximation to many in which first price mechanisms are used. There is serious reason for the users of these mechanisms to reconsider their choice.

Our conditions are most easily satisfied when the only source of asymmetry across bidders is via the handicap rule, or equivalently, if one player’s distribution over types is a constant shift of the other’s, a case we will argue is quite common in practice. But, we also make considerable headway when the underlying distributions over types differ across bidders.

A good entry point into the literature on asymmetric first price auctions is Maskin and Riley [36,37]. Our results comparing first and second price auctions are related to, and at first glance contradict, Maskin and Riley [36] and Kirkegaard [23] both of whom argue the superiority of first price auctions in a class of settings that intersects ours significantly. We discuss the connection in detail in Section 8.

Lebrun [25–27] characterizes equilibria in asymmetric first price auctions. The equilibrium in these auctions is unique under log-concave distributions (Maskin and Riley [38], Lebrun [28]). A key result on existence of monotone strategies in first price auctions is provided by Reny and Zamir [44].

---

4 Our measure of performance implies but is stronger than the standard ranking using the auctioneer’s ex-ante surplus
5 That is, in a complicated setting, the usual winner’s curse may be exacerbated by a bidder’s fear that winning may indicate having misunderstood the rules or erred in calculating a sensible bid.
6 For a further summary of the literature on asymmetric first price auctions, see Krishna [20] and the references therein.
Myerson [40] and Riley and Samuelson [46] begin a long discussion of implementation with asymmetric cost distributions. McAfee and McMillan [39] argue that one doesn’t want to always buy from the low-cost bidder. In Che [10], suppliers bid both a price $p$ and a quality $q$ which is evaluated via a quasi-linear scoring rule $S(p, q)$. The optimal such rule distorts quality downward, and can be implemented by either the first or second score rules. In their model, low costs are correlated with low quality. Our setting differs from these in that we assume that the buyer has a fixed and known relative valuation for purchasing from the two sellers. Laffont and Tirole [24] ask the more complex question of optimal favoritism when the auctioneer is an agent of the buyer.

Corns and Schotter [11] conduct experiments showing that small percentage bonuses yield a better average price in asymmetric environments. Shachat and Swarthout [47] provide further experimental evidence that the optimal English auction with a bonus outperforms a standard sealed bid request for proposal, in a setting with uniformly distributed costs. Cabral and Greenstein [7] discuss the empirical implications of favored bidding in federal procurement, while Wolfstetter and Lengweiler [54] analyze favoritism and corruption in the context of handicap auctions. Marion [34,35] estimates the price effect of favoring disadvantaged bidders through proportional bonuses and points out that bid preferences can have significant negative participation effects on non-favored bidders.

Section 2 presents the model, the basic equilibrium characterization, and the connection between equilibrium bid functions and the $\rho$-concavity of surplus functions. Section 3 discusses $\rho$-concavity and its implications for symmetric first price auctions. Section 4 returns to the analysis of asymmetric first price auctions and looks at the geometry of allocations when type densities are increasing. Section 5 uses the results on the geometry of allocations to compare first and second price mechanisms. Sections 6 and 7 extend the analysis. Section 9 concludes. The majority of proofs are in Appendix A.

2. Model and basic characterization

Consider a buyer seeking to procure a good from either or sellers 1 and 2 with costs $c_1$ and $c_2$. Costs are independent, from cumulatives $F_1$ and $F_2$, where $F_i$ has density $f_i$ which is twice continuously differentiable. Supports are finite and equal to $[\underline{c}_i, \overline{c}_i]$, where $[\underline{c}_2, \overline{c}_2]$ is normalized to $[0, 1]$. The reverse cumulative is $\overline{F}_i = 1 - F_i$, with $\frac{f_i}{\overline{F}_i}$ strictly increasing.

The buyer’s utility from purchasing the good at price $p$ from $i$ is $v_i - p$. Let $\Delta = v_1 - v_2 \geq 0$ be the amount by which 1 is preferred to 2.

---

7 Naegelen [41] extends Che’s result by allowing for an exogenous preference for one bidder. Branco [5] adds common value aspects and correlation to costs. Asker and Cantillion [3] expand the results to multi-dimensional quality and discuss the connection to scoring rules.

8 In Ganuza and Pechlivanos [15], the buyer, who chooses the design of the object to be procured, chooses a design which favors one firm, but then use the mechanism to “recapture” that advantage. If the buyer must use a symmetric mechanism, he chooses a design that increases homogeneity, and thus competition.

9 Flambard and Perrigne [13] provide empirical evidence from Canadian snow removal procurement auctions supporting this conclusion.

10 As Appendix B proves, the analysis carries over exactly to a standard auction setting with one seller and two buyers.

11 Given the normalization, we allow $\underline{c}_1 < 0$. This leads to no problems: shifting both distributions and the buyer’s value by a constant leaves our results unaffected.

12 Rezende [45] shows that in a setting where the buyer cannot commit to a mechanism, the buyer may prefer not to know $\Delta$. 

In a First Price Handicap Auction with handicap $A$ (FPHA$_A$) sellers 1 and 2 submit bids $b_1, b_2$. Seller 1 wins if and only if $b_1 < b_2 + A$. The winner is paid his bid. Seller $i$ is restricted to bid at most $\bar{c}_i$, so that bidders cannot receive a payment larger than their highest possible cost. We consider Bayesian equilibria in pure, continuous, and strictly increasing strategies $\beta_1$ and $\beta_2$, and in which $\beta_i(c) \geq c$ for all $c \in [\bar{c}_i, \hat{c}_i]$. Primitives for this can be found in Reny and Zamir [44] and Jackson and Swinkels [19].

Two other settings are closely related. In a First Price Bonus Auction with bonus $A$, 1 and 2 submit bids $b_1, b_2$. Seller 1 wins if and only if $b_1 < b_2$. If 2 wins, he receives $b_2$ while if 1 wins, he receives $b_1 + A$. Seller 1 is restricted to bid at most $\bar{c}_1 - A$. In a First Price Auction with Cost Shift $A$, seller 2 draws his cost from $F_2$, while seller 1 draws his cost from $F_{1,A}$ defined on $[\xi_1 - A, \bar{c}_1 - A]$ by $\bar{F}_{1,A}(c_1) = \bar{F}_1(c_1 + A)$. The low bidder wins and receives their bid, and 1 is again restricted to bid at most $\bar{c}_1 - A$.

These three settings are related in that bid functions form an equilibrium in one of these auctions if and only if their obvious translations are an equilibrium in the others. So, in studying auctions with handicaps, we are also studying auctions with bonuses and auctions in which type distributions differ in that one is shifted from the other by a constant.

Given FPHA$_A$, define the allocation function $\phi$ by

$$\beta_1(\phi(c_2)) = \beta_2(c_2) + A.$$ 

Seller 1 wins if and only if $c_1 < \phi(c_2)$.

Define $\psi$ as the inverse of $\phi$.

We now turn to a more detailed examination of the equilibrium bid and allocation functions. Assume in what follows that $\bar{c}_1 - A \leq 1$, so that bidder 2, for the range of costs $[\bar{c}_1 - A, 1]$ has no chance of winning. The case $\bar{c}_1 - A \geq 1$ is similar.

For an arbitrary positive function $g$, define

$$W_g(c) = \frac{gg''}{(g')^2}(c).$$

In the next section, we provide two useful interpretations of this object. For now, let us see that objects of the form $W_g$ are central to first price auctions.

**Theorem 1.** Assume that $\bar{c}_1 - A \leq 1$. Let $\beta_1, \beta_2$ be an equilibrium of FPHA$_A$. Then $\phi(0) = \xi_1$, and $\phi(\bar{c}_1 - A) = \hat{c}_1$. Surpluses with cost $c$ are

$$S_1(c) = \int_c^{\bar{c}_1} \bar{F}_2(\psi(s)) \, ds \quad \text{and} \quad S_2(c) = \int_c^{\bar{c}_1 - A} \bar{F}_1(\phi(s)) \, ds.$$  

If $f_1$ and $f_2$ are $C^k$, then $\beta_1, \beta_2$, and $\phi$ are $C^{k+1}[0, 1 - A]$. On their domains

$$\beta_1'(c) = W_{S_1}(c) > 0, \quad \beta_2'(c) = W_{S_2}(c) > 0.$$  

---

13 Ties are zero probability in equilibrium and the tie breaking rule is inessential (see Jackson and Swinkels [19]).

14 See Mares and Swinkels [29] for further details.

15 Since the equilibrium is strictly increasing, $\phi$ is well-defined and increasing.

16 Here and in a number of analogous situations that follow, it can be the case that for some $c_2$, $\beta_1(c_1) < \beta_2(c_2) + A$ for all $c_1$. In this event, define $\phi_{FP}(c_2) = \xi_1$. Similarly (although this will not in fact occur in this instance) if it were the case that for some $c_2$, $\beta_1(c_1) > \beta_2(c_2) + A$ for all $c_1$, one would define $\phi_{FP}(c_2) = \xi_1$. 

and

\[ \phi'(c) = \frac{\beta'_2(c)}{\beta_1'(\phi(c))} = \frac{S_1(\phi(c))}{S_2(c)} \frac{f_2(c)}{\bar{F}_2(\phi(c))} > 0. \]  

(3)

Unlike in symmetric auctions, \( S_1 \) and \( S_2 \) (and hence \( \beta_1 \) and \( \beta_2 \)) are not functions of primitives, but also depend on the entire behavior of \( \phi \) and \( \psi \) to the right of \( c \). This makes the analysis of asymmetric first price auctions difficult.

3. First price auctions and local concavity

Given Theorem 1, the analysis of the allocation of any given FPHA comes down to \( W_{S_1} \) and \( W_{S_2} \), as these are the slopes of the bid functions. As we shall see, the size and monotonicity of \( W_F, W_{\bar{F}}, \) and \( W_{\bar{F}} \) are also important in understanding first price auctions, optimal mechanisms and second price auctions. To understand these objects we begin in this section with a brief detour into \( \rho \)-concavity. We then investigate the links between results about \( \rho \)-concavity and equilibrium bidding in symmetric first price auctions.

It will turn out that for some of our results, it is relevant whether the \( \rho \)-concavity of specific functions is monotonically either increasing or decreasing. To help understand when this will hold and what it means, we do two things. First, we establish a direct connection between monotone \( \rho \)-concavity and the more familiar notion of monotone relative risk aversion for a utility function. Then we provide a closure result which has a simple geometric interpretation.

Let \( g \) be an arbitrary positive valued function \( g \) with support wlog \([0, 1]\). It is straightforward that \( \frac{g}{t} \) is concave at \( c \) if and only if \( t \leq 1 - W_g(c) \). The local \( \rho \)-concavity of \( g \) at \( c \) is thus defined as

\[ \rho_g(c) \equiv 1 - W_g(c). \]

Standard concavity is thus equivalent to local \( \rho \)-concavity everywhere at least 1, while log-concavity is equivalent to local \( \rho \)-concavity everywhere at least 0.

Let \( G(c) = \int_c^1 g(s) \, ds \), and let \( \bar{G}(c) = \int_c^1 g(s) \, ds \). Then, we have the following extension of Prekopa [42,43] and Borell [4].

**Theorem 2.** For any \( c \) where \( g'(c) > 0 \),

\[ \frac{\max_{[0,c]} \rho_g(s)}{1 + \max_{[0,c]} \rho_g(s)} \geq \rho_G(c) \geq \frac{\min_{[0,c]} \rho_g(s)}{1 + \min_{[0,c]} \rho_g(s)}. \]  

(4)

For any \( c \) where \( g'(c) < 0 \),

\[ \frac{\max_{[c,1]} \rho_g(s)}{1 + \max_{[c,1]} \rho_g(s)} \geq \rho_{\bar{G}}(c) \geq \frac{\min_{[c,1]} \rho_g(s)}{1 + \min_{[c,1]} \rho_g(s)}. \]  

(5)

17 We will assume that any \( g \) we deal with is sufficiently well behaved that \( W_g(1) = \lim_{x \to 1} W_g(x) \) is well defined in the extended real line. We will also assume that when \( g(1) = 0, W_g(1) \) is finite. This last condition is a very mild: see Lemma 12 in Appendix A for a discussion.
Anderson and Renault [1] obtain a very similar result in studying Cournot and other oligopoly settings. The fact that they find $\rho$-concavity (in their case of a demand function) at the heart of their results strengthens our belief in the importance of this approach to auction theory, since along the lines of Bulow and Roberts [6], there is a strong relationship between a demand curve in a monopoly problem and a type distribution in a mechanism setting.

A first easy application of Theorem 2 lets us look at how cost changes are shared between the seller and the buyer in a symmetric setting.

**Proposition 1.** Consider a symmetric standard FPA with $F_1 = F_2 = F$ and $A = 0$. If $f$ is increasing, then $\beta'(c) \leq \frac{1}{2}$ for all $c$. If $f$ is decreasing, then $\beta'(c) \geq \frac{1}{2}$ for all $c$.

To see this, note that for the symmetric case, $S_1(c) = S_2(c) = \frac{1}{2} \bar{F}(s) ds$, and so by Theorem 1 $\beta'(c) = W_\frac{1}{2} \bar{F}(c) = 1 - \rho_\frac{1}{2} \bar{F}(c)$. If $f$ is increasing, then $\bar{F}$ is concave and so $\rho_\frac{1}{2} \bar{F}$ is everywhere at least 1. Thus by Theorem 2,

$$W_\frac{1}{2} \bar{F} = 1 - \rho_\frac{1}{2} \bar{F} \leq \frac{1}{2}.$$  

Similarly, if $f$ is decreasing, then $\rho_\frac{1}{2} \bar{F} \leq 1$, from which $\rho_\frac{1}{2} \bar{F} \leq \frac{1}{2}$, and so $\beta'(c) \geq \frac{1}{2}$.

Consider two symmetric first price auctions with cost distributions $F$ and $G$, and with equilibrium bid functions $\beta_F(c)$ and $\beta_G(c)$. Under what conditions can one say that bidding is more or less aggressive with $G$ than with $F$? A partial answer follows.

**Proposition 2.** Let $F$ with support $[a, 1]$, $a \geq 0$, and $G$ with support $[0, 1]$ be related by $G(c) = F(\gamma(c))$, where $\gamma : [0, 1] \rightarrow [a, 1]$ satisfies $\gamma(c) \geq c$.

1. If $W_\frac{1}{2} \bar{F}$ is decreasing\(^{19}\) and $\gamma$ is convex then for all $c \in [a, 1]$,

   $$\beta'_G(c) \leq \beta'_F(c) \quad \text{and} \quad \beta_G(c) \geq \beta_F(c).$$

2. If $W_\frac{1}{2} \bar{F}$ is increasing and $\gamma$ is concave then for all $c \in [a, 1]$,

   $$\beta'_G(c) \geq \beta'_F(c) \quad \text{and} \quad \beta_G(c) \leq \beta_F(c).$$

One way of thinking about $F$ is that one first draws a cost according to $G$, and then suffers a cost penalty given by $\gamma(c) - c$. Concavity of $\gamma$ says that incremental cost increases in the original draw have lower incremental impact on final costs when costs are higher than when they are lower, and similarly for convexity.\(^{20,21}\)

---

\(^{18}\) Our result differs from Anderson and Renault primarily in that we can restrict ourselves to the maximum and minimum $\rho$-concavity to one side of the point in question, and that we do not need monotonicity of $g$. Weyl and Fabinger [52] also use a similar result.

\(^{19}\) Examples for which this is true include $f(x) = \frac{3}{2} - x$, $f(x) = 1 - \sqrt{x}$, and $f(x) = e^{-x}$. See the next section for further discussion.

\(^{20}\) When $\gamma$ is concave, $G$ dominates $F$ in the convex transform order, and conversely when $\gamma$ is convex. See Shaked and Shanthikumar [48]. See Ganuza and Penalva [16] for an auction application.

\(^{21}\) Hopkins [17] and Hopkins and Kornienko [18] also obtain rankings of bids across distributions using simple tools. The analysis based on $\rho$-concavity allows us to rank the slopes of the bidding functions.
Under part (1) of the proposition, while costs are stochastically lower under \(G\), bids for any given cost are higher, and less of any given cost saving for a bidder shows up in a more aggressive bid.\(^{22}\) Under part (2), the buyer benefits both from the stochastically lower costs implied by \(G\) and from more aggressive bidding.

The proof follows from (2) combined with the following observation about \(\rho\)-concavity, and the observation that \(\beta_G(1) = \beta_F(1) = 1\).

**Lemma 1.** If \(W_{\bar{F}}\) is increasing (decreasing), \(\gamma\) is concave (convex), and \(\gamma(c) \geq c\) for all \(c \in [a, 1]\), then \(W_{\bar{F}} \circ \gamma(c) \geq (\leq) W_{\bar{F}}(c)\) for all \(c \in [a, 1]\).

As a final quick application, let us depart momentarily from the two bidder setting.

**Proposition 3.** Fix \(F\), and let \(\beta_n\) be the symmetric first price equilibrium bid function with \(n\) bidders. Then \(\beta_n'(c)\) increases in \(n\) and \(\beta_n(c)\) decreases in \(n\) for all \(c\).

That \(\beta_n(c)\) is decreasing in \(n\) is well known, as \(\beta_n(c)\) is the expectation of the lowest cost from \(n-1\) bidders. But, as Proposition 2 illustrates, how incremental cost improvements are shared is less obvious. The proof relies on the fact that, as for the two bidder case, \(\beta_n'(c) = 1 - \rho \int_{\bar{F}_{n-1}}(c)\).

The result is then implied by the following property of \(\rho\)-concavity, and by the observation that \(\beta_n'(1) = 1\) for all \(n\).

**Lemma 2.** Let \(g\) be decreasing and log-concave and \(\alpha > 1\). Then \(\rho_{\int g^n}(c) \leq \rho_{\int g}(c)\).

In particular, to compare \(\beta_{n+1}'(c)\) and \(\beta_n'(c)\), set \(\alpha = \frac{n}{n-1}\) and \(g(c) = \bar{F}^{n-1}(c)\).

### 3.1. Monotone local concavity

In this section, we consider the question of whether the local curvature for a given \(g\) increases as we move along its support. That is, when will \(W_g\) be monotone? This is relevant to the interpretation of Proposition 2, and will be of first importance at several other points as we move forward. For example, monotonicity of \(W_F\) determines the shape of virtual costs, and hence the slope of the optimal allocation (see Section 5.2 below).

We begin by relating monotonicity of \(W_g\) to a much more familiar concept. Recall that for a function \(u\), the coefficient of relative risk aversion of \(u\) is \(ru(u) = -\frac{u''(w)w}{u'(w)}\).

**Proposition 4.** For \(g\) monotone, let \(u = g^{-1}\). Then, for each \(c\), \(W_g(c) = ru(g(c))\). Thus, for \(g\) increasing, \(W_g\) and \(ru\) have the same monotonicity, and for \(g\) decreasing, \(W_g\) and \(ru\) have opposite monotonicity.\(^{23,24}\)

---

\(^{22}\) The buyer is better off under \(G\) than under \(F\), with the effect through better costs dominating the effect through less aggressive bidding.

\(^{23}\) If \(g\) is decreasing, one might for interpretational reasons define \(u^{-1}(1-c) = g(c)\), so that \(u\) is increasing. It is immediate that \(ru\) is invariant to this choice.

\(^{24}\) To see the proof, differentiate the identity \(u(g(c)) = c\) twice to arrive at

\[u''(g(c))(g'(c))^2 + u'(g(c))g''(c) = 0.\]
So, the concept of monotone local concavity is identical to the concept of monotone relative risk aversion. An implication is that the class of functions with the various properties we utilize is rich.

**Example 1.** Consider the hyperbolic absolute risk aversion utility functions

\[ u(w) = \frac{\gamma}{1 - \gamma} \left[ \frac{1}{\alpha} + \frac{w}{\gamma} \right]^{1 - \gamma}, \]

for any \( \alpha \neq 0 \) and \( \gamma \notin \{0, 1\} \), where the domain of \( w \) is an interval \([w, \bar{w}]\) where \( \frac{1}{\alpha} + \frac{w}{\gamma} > 0 \).

Then, \( r_u(w) \) is increasing if \( \alpha > 0 \) and decreasing if \( \alpha < 0 \), and so \( g(u) = u^{-1} \) has increasing if \( \alpha > 0 \) and decreasing otherwise. Anywhere that \( ru \leq 1 \), \( g \) is log-concave.

By adding a constant to \( u(w) \), we have \( g \geq 0 \). If we define \( \hat{u}(w) = u(w) - u(\bar{w}) / u(\bar{w}) - u(w) \), then \( g \) is a cumulative on \([\hat{u}(w), \hat{u}(\bar{w})]\).

Both the intuition for monotonicity and the ability to generate rich classes of examples are enhanced by the following lemma, which states that if \( g \) is zero at one end or the other and has monotone local concavity, then its integral from that end inherits the same monotonicity.

**Lemma 3.** If \( g(0) = 0 \) and \( \rho_g \) is monotone on some interval \([0, \hat{c}]\), then, \( \rho \int_0^{\hat{c}} g(s) ds \) and \( \rho g(\hat{c}) \) share the same monotonicity on \([0, \hat{c}]\). If \( g(1) = 0 \), and \( \rho_g \) is monotone on \([\hat{c}, 1]\), then \( \rho \int_{\hat{c}}^1 g(s) ds \) and \( \rho_g \) share the same monotonicity on \([\hat{c}, 1]\).

**4. The geometry of allocations**

We turn now to the central question of characterizing the slope of the allocation \( \phi \) for a given FPHA. Our approach is essentially geometric in that we first establish a connection between the curvature of \( \phi \) at specific points and the underlying \( \rho \)-concavity properties of the fundamentals \( \bar{F}_1 \) and \( \bar{F}_2 \). Then, we will further tie down the geometry of \( \phi \) by investigating its properties at the boundaries. These results will be central when we look at the relative merits of first and second price mechanisms in Section 5.

**Theorem 3.** Let \( r \) be an interior global minimum of \( \phi' \). Assume that \( \phi'(r) \leq 1 \), that \( \liminf_{c \to \hat{c}^-} \phi'(c) > 1 \), and that \( \frac{f_1}{F_1}(\phi(r)) - \frac{f_2}{F_2}(r) \geq 0 \). Let

\[
H(r) = \left( \frac{1}{W_{\bar{F}_1}(\phi(r))} - 2 \right) \left( \frac{f_1}{F_1}(\phi(r)) - \frac{f_2}{F_2}(r) \right) + \frac{f_2^2}{f_2}(r) - \frac{f_1^2}{f_1}(\phi(r)).
\]

If \( H(r) \geq 0 \), then \( \phi''(r) > 0 \).

Rearrange and multiply both sides by \( g(c) \) to arrive at

\[
r_u(g(c)) = \frac{-u''(g(c))g(c)}{u'(g(c))} = \frac{g(c)g''(c)}{u'(g(c))^2} = W_g(c).
\]

**25** Taking \( \gamma \to 1 \) gives log utility, \( \gamma \to -1 \) quadratic utility and \( \gamma \to \infty \) negative exponential utility.

**26** Note that \( g'(u(w)) = u'(w) \), and so \( g'(u(w)) / g(u(w)) = 1 / uw(w) \), which for \( u \) increasing is increasing if and only if \( ru \leq 1 \).
The power of Theorem 3 is that $H$ does not depend on the details of the behavior of $\phi$ after $r$, but only on the primitives of the problem. The proof begins from (3) in Theorem 1. This expression involves the surpluses $S_1(\phi(r))$ and $S_2(r)$ which as functions of the entire equilibrium are inherently forbidding. Central to the proof is that by a change of variables,

$$S_2(r) = \int_r^{\bar{c}_1-A} \bar{F}_1(\phi(s)) \, ds$$
$$= \int_{\phi(r)}^{\phi(\bar{c}_1-A)} \bar{F}_1(s) \psi'(s) \, ds$$
$$< \frac{1}{\phi'(r)} \int_{\phi(r)}^{\bar{c}_1} \bar{F}_1(s) \, ds,$$

where the inequality follows because $r$ is a global minimum of $\phi'$. We have thus replaced $S_2(r)$ with a much simpler object, one which can be analyzed using the tools of $\rho$-concavity discussed above.

4.1. The geometry of allocations for symmetric (or shifted) costs

In this section, we specialize our model to one that is simple but highly relevant. From this point on, we assume that $f$ is log-concave. More importantly, we consider the case $F_1 = F_2 = F$. One way of thinking of this is that the only remaining asymmetry in the model is in the preferences of the buyer as expressed in $A$. But, recalling the isomorphism between auction forms discussed above, another setting to which this applies is when the buyer has no preferred supplier, but the costs of the two suppliers have distributions that are shifted by a constant.27

As an example, consider a buyer contracting for a bulk good from two suppliers who differ only in their distance from the buyer, or of on-site professional services again from suppliers at differing distances. Then, it is reasonable to think of costs before delivery as symmetrically distributed, but transportation costs as differing by a fixed known amount. The question then is to what degree the buyer should optimally subsidize the extra transportation costs of the more distant supplier. In settings like these, either first or second price mechanisms that treat the players differently are particularly easy to achieve. Rather than naming one bidder as “favored”, the mechanism can treat both players according to the same rules, but specify what part of transportation costs are the responsibility of the buyer.

As another example, consider a firm that needs to replace a key piece of software. One vendor supplies a product compatible with the existing capabilities of the firm, while another provides a product that requires new hardware or training. By writing the rules of the auction to specify the scope of the project, the costs of the hardware or training can be reflected to varying degrees in how the two bidders compete. In this light, Cabral and Greenstein [7] analyze federal procurement auctions for computers where incumbents are favored through a fixed handicap generated by the General Accounting Office to reflect an estimate of the costs of moving to a new system. Intriguingly, this handicap is routinely felt by the department involved to be too low, something which Cabral and Greenstein argue, and our results support, can be optimal from an ex-ante point of view. Wolfstetter and Lengweiler [54] present additional examples of handicaps and bonuses implemented via scoring rules.

27 We relax the assumption that costs are symmetric (or shifted) later.
We begin with some simple observations about $\phi$.

**Lemma 4.** Let $F_1 = F_2 = F$, and let $A \geq 0$. Then,

1. $\phi'(0) > 1$,
2. $\lim \inf_{c \to 1-A} \phi'(c) > 1$, and
3. $\phi(c) \in (c, c + A)$ for all $c \in (0, 1 - A)$.

The first two parts say that $\phi' > 1$ at each boundary. The third part says that the favored bidder takes part of his advantage in the form of a higher margin, thus undoing some of the distortion designed by the auctioneer, and part in the form of a higher probability of winning.\(^{28}\)

This in hand, we have our first application of Theorem 3. We show that if $f$ is weakly increasing, then the distortion induced by $FPHA_A$, $\phi(c) - c$, is monotonic. We relax the assumptions that $f$ is increasing later.

**Proposition 5.** Assume that $F_1 = F_2 = F$, and that $f$ is weakly increasing. Then, $\phi'(c_2) > 1$ for all $c_2$.

The proof begins by observing that by Lemma 4, if the proposition is false then $\phi'$ has an interior global minimum $r$ with $\phi'(r) \leq 1$. We show that all parts of $H$ in Theorem 3 are then positive, contradicting that $\phi'$ was minimized at $r$.

One application of this proposition is in combination with the following result.

**Lemma 5.** Assume that $F_1 = F_2$. If $\phi' \geq 1$ everywhere, then $\beta_1$ and $\beta_2$ lie on either side of the symmetric equilibrium strategy $\beta_s$ of a standard first price auction. That is,

$$\beta_1 \geq \beta_s \geq \beta_2.$$

Thus, 1 bids less aggressively than if he were not favored, and 2 more aggressively. Fig. 1 shows, based on a numerical solution, how $\beta_1$ and $\beta_2$ vary in $A$ for the uniform case. It is an interesting open question whether $\beta_1$ and $\beta_2$ move monotonically further apart as $A$ grows for general $f$.\(^{29}\)

### 5. Ranking first and second price handicap auctions

We now turn to another implication of Proposition 5. We identify conditions under which we can compare the merits of first and second price mechanisms. We do this by comparing how each mechanism compares to the mechanism which is optimal subject to always allocating the job (that is, we compare each mechanism to the optimal non-excluding mechanism).

---

\(^{28}\) This lemma is subsumed by Lemmas 15 and 18 (in Appendix A) which apply in a more general context. We recommend deferring those proofs to after the more general setting has been introduced. For the impatient reader, in the case of current interest the object $\gamma_f$ used in the proof is the identity and the object $\tau$ is 0.

\(^{29}\) For an excellent introduction to numerical techniques for asymmetric first price auctions see Marshall et al. [32] and Marshall and Schulenberg [33].
Fig. 1. The equilibrium bid functions for $A = 0$ (dotted), $A = .2$ (dashed) and $A = .4$ (solid) for uniformly distributed costs.

5.1. Second price mechanisms

In a Second Price Bonus Auction with bonus $A$ (SPBA$_A$) the auctioneer announces a bonus $A$, and requests sealed bids from 1 and 2. The low bidder wins (as before, ties are inessential). When 1 wins, he receives $\min(b_2 + A, \bar{c}_1)$, while if 2 wins, he receives $\min(b_1, 1)$. Putting these maxima on payments guarantees that the bidders do not receive more than their highest possible cost. It is weakly dominant for 2 to set $\beta_2(c_2) = c_2$, and 1 to set $\beta_1(c_1) = c_1 - A$. Thus, 1 wins if and only if $A \geq c_1 - c_2$. Given this, 2 never wins when $c_2 > \bar{c}_1 - A$. So, without loss of generality, we restrict $A \leq \bar{c}_1$.

An SPBA can always be replicated by an open mechanism with bonuses. Thus all of our results about SPBAs can be directly translated into statements about open auctions.

5.2. Optimal mechanisms

Let

$$\omega_i(c_i) = c_i + \frac{F_i(c_i)}{f_i(c_i)}$$

be the virtual cost of $i$. Note that $\omega'_i = 1 + \rho F_i$, giving another simple link to $\rho$-concavity. For example $\omega_i$ will be concave or convex depending on whether $\rho F_i$ is decreasing or increasing (recall Section 3.1). Because $F_i$ is log-concave, $\omega'_i \geq 1$.

---

30 To see this, imagine an open descending price mechanism where bidders choose when to drop out and the last active bidder wins, where if 2 wins, he receives the prevailing price, while if 1 wins, he receives the prevailing price plus a bonus $A$, and where we start the clock at $\bar{c}_1 - A$. It is weakly dominant for 2 to drop out at $c_2$ and for 1 to drop out at $c_1 - A$.

31 While log concavity, or equivalently $\rho F_i > 0$, is the conventional assumption in a large part of the auction literature (partly on the basis that it is interpretable), the implementability condition that $\omega'_i > 0$ is precisely $\rho F_i > -1$, i.e., that $\frac{1}{F_i}$ is convex.
Consider any deterministic non-excluding mechanism — one in which the buyer always buys. From incentive compatibility, the mechanism is characterized by an increasing function \( \eta \) such that 1 wins if and only if \( c_1 < \eta(c_2) \). Let \( \eta_M \) be the optimal non-excluding mechanism. For any two non-excluding mechanisms \( \eta_1 \) and \( \eta_2 \), say that \( \eta_1 \) dominates \( \eta_2 \) if \( \eta_1 \) and \( \eta_M \) generate the same allocation on a superset of the set over which \( \eta_2 \) and \( \eta_M \) do.

**Lemma 6.** The optimal non-excluding mechanism, \( \eta_M \), is given by

\[
\Delta = \omega_1(\eta_M(c_2)) - \omega_2(c_2).
\]

If \( \eta_1 \) dominates \( \eta_2 \), then \( \eta_1 \) gives the buyer higher surplus than \( \eta_2 \).

This follows along the lines of Myerson [40] or Riley and Samuelson [46].

**Lemma 7.** If \( F_1 = F_2 = F \) and \( \rho_F \) is increasing, then \( \eta'_M \leq 1 \). If \( \rho_F \) is decreasing, then \( \eta'_M \geq 1 \).

That is, when cost distributions are symmetric then the optimal mechanism has slope greater or less than 1 depending on whether virtual costs are concave or convex. To see this, note from (7) that since \( \omega \) is increasing \( \eta_M(c_2) \geq c_2 \), and that

\[
\eta'_M(c_2) = \frac{1 + \rho_F(c_2)}{1 + \rho_F(\eta_M(c_2))}.
\]

Note also using (9) that each of the three conditions \( \eta'_M = 1, \rho_F \) constant, and the optimal mechanism being implementable by an appropriate SPBA are equivalent. This is true for the power distributions, \( F(c) = c^\alpha \), where it is easily calculated that the optimal SPBA has bonus \( A = \frac{\alpha}{1+\alpha} \Delta \).

### 5.3. A ranking result

Despite the strength of our dominance criterion, our results allow us to show simple conditions under which, for any FPHA, a well chosen second price mechanism dominates it.

**Theorem 4.** Assume that \( F_1 = F_2 = F \), that \( f \) is increasing, and that \( \rho_F \) is increasing. Then, for any \( A \), there is \( \hat{A} \) such that SPBA \( \hat{A} \) strictly dominates FPHA \( A \).

To see the proof, consider Fig. 2. From Lemma 7, \( \eta'_M \leq 1 \) everywhere, while from (7), \( \eta_M \) lies above the diagonal. By Proposition 5, \( \phi' > 1 \) everywhere, and \( \phi(0) = 0 \). Hence, either \( \phi \) and \( \eta_M \) cross as illustrated, or \( \phi \) lies everywhere below \( \eta_M \). In the first case, choose \( \hat{A} \) so that SPBA \( \hat{A} \)

---

32 Exclusion may or may not be optimal. But, in either case, the optimal constrained mechanism serves its purpose in helping us compare the FPHA and SPBA.

33 To see this, note from (7) that since \( \omega \) is increasing \( \eta_M(c_2) \geq c_2 \), and that

\[
\eta'_M(c_2) = \frac{1 + \rho_F(c_2)}{1 + \rho_F(\eta_M(c_2))}
\]

yielding the result.

34 Gabaix [14] argues that such distributions are common in economic settings.
generates the allocation given by the dotted line in Fig. 2. This agrees with the optimal allocation strictly more often than does FPHA\(_A\). In the second case, the same is true of any \(\hat{A}\) giving an allocation that lies above \(\phi\) and below \(\eta_M\).

**Corollary 1.** Assume that \(F_1 = F_2 = F\), that \(f\) is increasing, and that \(\rho_F\) is increasing. Then, the optimal SPBA has strictly higher expected surplus than any FPHA.

To see this, note that the optimal FPHA is strictly dominated by some SPBA, and so *a fortiori* has lower expected surplus to the buyer than the optimal SPBA. Thus, an auctioneer contemplating using a handicap auction of either the first price or second price format can restrict his search to second price mechanisms and know he has not lost anything.\(^{35}\)

**Example 2.** Let \(g\) be any function for which \(W_g\) is decreasing, or, equivalently, for which \(g^{-1}\) has decreasing relative risk aversion. Letting \(f(c) = \int_0^c g(s) \, ds\), \(f(0) = 0\), and \(W_f\) is decreasing. By Lemma 3, it follows that \(W_f\) is decreasing as well.\(^{36}\) and thus \(f\) satisfies the conditions of Theorem 4.

Under the conditions of Proposition 4, for any given first price handicap mechanism, there is a superior second price mechanism. Our conditions are sufficient for this result, but far from necessary. In Mares and Swinkels [31], we show how to extend the result to a setting in which there are several players with costs drawn according to each of two distributions. One can also numerically calculate equilibria of two player asymmetric first price auctions, and show that the dominance result continues to hold in a wide variety of other examples. In particular, we have no counterexample to \(\phi' > 1\). In Mares and Swinkels [30] we show how bounds on \(\rho_F\) translate into recommendations for second price handicaps that do not depend on further details of \(F\), but

\(^{35}\) Note also that there is no claim that for any given handicap \(\hat{A}\), SPBA\(_\hat{A}\) dominates FPHA\(_\hat{A}\). This is relevant to our comparison of our results to Maskin and Riley [37] (see Section 8).

\(^{36}\) Our class of examples is in one sense large (one can start from any \(g\) of the form described and construct an \(f\)), but on the other hand by construction always has \(f(0) = 0\), and so is also in a sense restrictive. It is easily shown that if \(\rho_F\) is increasing and \(f\) increasing, then \(f(0)\) must be 0.
guarantee surprisingly good performance even in settings where our dominance results do not apply.

6. Asymmetric costs

We now turn to a situation in which costs are asymmetric, to explore the robustness of Theorem 4, and to illustrate on a technical level what is involved.\(^{37}\) We begin with the case of asymmetric increasing densities. In the next section, we relax the monotonicity assumption.

To relate asymmetric cost distributions, it is useful, as we did in Proposition 2, to define \(\gamma\) implicitly by

\[
\bar{F}_1(\gamma(c_2)) = \bar{F}_2(c_2).
\]

Define \(\lambda_A\) by \(\lambda_A(c_2) = c_2 + A\), so that SPBA\(_A\) implements precisely \(\lambda_A\). Of particular interest to us will be \(\lambda_\tau\), where \(\tau = \bar{c}_1 - 1\). This is the line of slope 1 that passes through \((\bar{c}_1, 1)\). See Fig. 3.

Our first assumption is that \(F_1\) is, in a particular sense, a stretch and convexification of \(F_2\).

**Assumption 1.** \(\gamma' \geq 1\) and \(\gamma'' \geq 0\), with \(\gamma'(1)\) finite.\(^{38,39}\)

This is, of course, automatic when \(F_1 = F_2\). An implication of A1 is that \(\gamma \leq \lambda_\tau\). In Example 2 in Maskin and Riley [36], a “strong” buyer has values which are a linear stretch around 0 of those of a “weak” buyer, and the first price auction is superior. Our class of asymmetric auctions

---

\(^{37}\) Several different extensions in this direction are feasible.

\(^{38}\) That is, \(F_1\) dominates \(F_2\) in the dispersive order. See Shaked and Shanthikumar [48]. For application of the dispersive order to auction theory, see Ganuza and Penalva [16].

\(^{39}\) There is no assumption here that for example \(\gamma(0) \geq 0\). So, it can be that \(c_1\) does not first order stochastically dominate \(c_2\).
consists of those where \( F_1 \) can be viewed as a convexification and stretch of \( F_2 \), and so includes this case.\(^{40}\)

We remain in the case where virtual costs are convex.

**Assumption 2.** \( \rho_{F_1} \) is increasing.

**Theorem 5.** Assume \( A_1 \) and \( A_2 \), that \( f_1 \) is increasing, and that \( \Delta > \omega_1(\tilde{c}_1) - \omega_2(1) \). Then, for any \( A \), there is \( \hat{A} \) such that SPBA\(_\hat{A}\) dominates FPHA\(_A\).

The assumption \( \Delta > \omega_1(\tilde{c}_1) - \omega_2(1) \) says that \( \Delta \) is large enough that when both players have their worst cost types, the optimal non-excluding mechanism gives the job to 1. This is automatic for \( \Delta > 0 \) when \( F_1 = F_2 \), and otherwise requires that \( \Delta \) be sufficiently large compared to the asymmetry between \( F_1 \) and \( F_2 \).

As before, the proof of Theorem 5 is essentially geometric. See Fig. 3 which generalizes Fig. 2. First, we show that because \( \Delta > \omega_1(\tilde{c}_1) - \omega_2(1) \), \( \eta_M \) lies above \( \lambda_T \) and has slope at most 1. But then, for any \( A \leq \tau \), FPHA\(_A\) is dominated by SPHA\(_\tau\). The proof (see Appendix A) uses Theorem 3 to show that for \( A > \tau \), \( \phi'(c_2) \geq 1 \) over all relevant ranges. But then, an SPBA which implements a line of slope 1 through the unique crossing point of \( \eta_M(c_2) \) and \( \phi(c_2) \) dominates FPHA\(_A\). This is \( \lambda_{\hat{A}} \) as illustrated in Fig. 3.

7. Other densities

At the heart of our proof that \( \phi' > 1 \) is the expression (6). For increasing densities determining the sign of this expression was a relatively easy task, since all the terms involved are positive. In extending the results to hump-shaped or decreasing densities we face several difficulties, since the leading term in (6) is no longer positive. In exchange for abandoning the assumption that \( f_1 \) is increasing, we need two assumptions. Each is true when \( f_1 \) is increasing, but much more generally as well.

**Assumption 3.** \( \rho_{\bar{F}_1} \) is minimized at \( \tilde{c}_1 \).

Note first that if \( \rho_{\bar{F}_1} \) is decreasing then \( A_3 \) is automatic.\(^{41}\) Further, observe that by Remark 1 (see Appendix A.2), this is automatic if \( f_1 \) is increasing. On the other hand, \( A_3 \) can never be satisfied if \( f_1 \) decreases near \( \tilde{c}_1 \) but \( f_1(\tilde{c}_1) > 0 \).\(^{42}\) Thus we are restricted to densities where \( f_1(\tilde{c}_1) = 0 \). As we argue below, for such densities, the condition is neither vacuous nor difficult to satisfy.\(^{43}\)

---

\(^{40}\) As we shall see in Section 8, we will have the “contradictory” result that when handicaps can be chosen, a second price mechanism is preferred. The general “stretch” used by Maskin and Riley is somewhat different than ours, but coincides for this example.

\(^{41}\) Weyl and Fabinger [52] require an analogous monotonicity condition for their results. They observe that the condition is satisfied by many standard distributions used in econometric estimation.

\(^{42}\) If \( f \) is decreasing on \( (\tilde{c}, \tilde{c}_1) \), then \( W_{\bar{F}} > 0 \) on \( (\tilde{c}, \tilde{c}_1) \). If \( f'(\tilde{c}_1) > 0 \), but \( f'(\tilde{c}_1) \) is finite, then, since \( \frac{\bar{F}_1}{f_1}(\tilde{c}_1) = 0 \), \( W_{\bar{F}_1}(\tilde{c}_1) = 0 \), hence \( \rho_{\bar{F}_1} \) cannot be minimized at \( \tilde{c}_1 \).

\(^{43}\) That \( A_2 \) and \( A_3 \) are consistent is easily shown by example. One reason why the two assumptions can easily both be satisfied is that since \( f \) is log-concave, \( \rho_F - \rho_{\bar{F}} \) is increasing. This follows directly from the definitions using that \( \bar{F} = 1 - F \).
Our second assumption requires that in a sense the concavity of log $f$ versus that of log $\tilde{F}$ is at its smallest at $\tilde{c}_1$.

Assumption 4. \(\frac{(\log f_1)''}{(\log F_1)''}\) is minimized at $\tilde{c}_1$.\(^{44}\)

This is satisfied broadly but not universally in examples we have checked. It is a simple exercise that $A2$–$A4$ are satisfied for $F_1(c) = F_2(c) = 1 - (1-c)^\alpha$. The next result provides a ready source of both hump-shaped and decreasing examples.

Lemma 8. Let $G$ satisfy $A3$ and $A4$, and let $F_1$ be given by $\tilde{F}_1 = \tilde{G}^n$, for some $n \in (0, \infty)$. Then, $\tilde{F}_1$ satisfies $A3$ and $A4$ as well.

When $n$ is an integer greater than one, $\tilde{F}_1$ represents the most favorable from $n$ draws from $\tilde{G}$, while when $\frac{1}{n}$ is an integer, $\tilde{G}$ represents the most favorable from $n$ draws from $\tilde{F}_1$.

Choose $g$ increasing and log-concave with $W_g(\tilde{c}_1)$ finite. Then, $W_{\tilde{G}}$ is negative and increasing, so that $A3$ is satisfied, and $A4$ is automatic.\(^{45}\) In the context of Lemma 8 it is easily checked that for $n$ near enough 1, the resulting $f_1$ will thus be hump-shaped, while for $n$ large enough, $f_1$ will be decreasing.

In the same spirit, starting from $g$ increasing and log-concave, consider $f_1 \propto \tilde{G}$. Then by Remark 1 $W_{f_1} = W_{\tilde{G}}$ is increasing, and thus, by Lemma 3, so is $W_{\tilde{F}_1}$. This generates a class of decreasing densities which satisfy $A3$. It is easy to check that densities constructed this way satisfy $A4$ as well. Finally, by judicious choice of $\Delta$, one can start from symmetric distributions that satisfy the conditions, and distort one or the other of them via $\gamma$ in such a way as to continue to satisfy them.

Another issue we must face when $f_1(\tilde{c}_1) = 0$ is that over a range near 1, the slope of the optimal allocation function, $\eta_M'$, will no longer be less than 1.

Lemma 9. Assume $A1$, that $f_1(\tilde{c}_1) = 0$, and that $\gamma$ is not the identity. Then, for any finite $\Delta$, $\eta_M(1) = \tilde{c}_1$. Further, $\eta_M$ crosses $\gamma$ at some unique $c_\Delta \in (0, 1)$. As $\Delta$ increases, so does $c_\Delta$, and for $\Delta \to \infty$, $c_\Delta \to 1$. On some interval near 1, $\eta_M'(c) > 1$.

The problem is that virtual costs diverge as costs increase, and since near 1, $\gamma' > 1$ (as $\gamma$ is not the identity) they diverge faster for player 1, eventually swamping any given $\Delta$.

We thus have the possibility illustrated in Fig. 4. For clarity, we have zoomed in on the top right corner of the picture, starting at some cost $K$ for 2 and $\eta_M(K)$ for 1. Here, the SPBA through the intersection of $\eta_M$ and $\phi$ does not dominate $\phi$, because on the shaded triangular region, $\phi$ is getting things right while $\lambda_\tilde{A}$ is not.\(^{46}\) The key is that $\lambda_\tau$ no longer lies everywhere above $\eta_M$. It can be shown, however, that as either $\Delta$ grows or $F_1$ and $F_2$ become close to each other, the region of concern becomes arbitrarily small.

To attain a result in this setting, we relax our notion of one mechanism being better than another from dominance to simple ex-ante superiority. We will also need a different form of our

\(^{44}\) The expression is well-defined at $\tilde{c}_1$. See Lemma 20 in Appendix A.

\(^{45}\) By log-concavity $\frac{(\log g)''}{(\log G)''}$ is weakly positive. Since $g$ is increasing, $W_{\tilde{G}}(\tilde{c}_1) = 0$ and thus by Lemma 20 the expression converges to 0 at $\tilde{c}_1$.

\(^{46}\) Since virtual costs diverge in this case, some exclusion is optimal. We leave the question of the effects of such exclusion for future research.
Fig. 4. Illustration of difficulty when $f_1(\bar{c}_1) = 0$.

Fig. 5. Illustration of $\delta$ (thick line) when $c_\tau < c^*_2$.

Assumption that $\Delta$ is not too small relative to the asymmetry between $F_1$ and $F_2$. To do this, define $c_\tau$ as the (unique) point at which $\eta_M$ crosses $\lambda_\tau$, and consider the line $\delta$ given by

$$
\delta(c) = \begin{cases} 
\lambda_\tau(c) & \text{for all } c \leq c_\tau, \\
\eta_M(c) & \text{for all } c \geq c_\tau,
\end{cases}
$$

which is illustrated in Fig. 5.

**Assumption 5.** There exists $\hat{A}$ such that $BS(\lambda_{\hat{A}}) \geq BS(\delta)$. 
That is, there is a second price auction that does at least as well, in ex-ante terms, as \( \delta \). This assumption can again be interpreted as saying that \( /Delta_1 \) is not too small relative to the asymmetry between \( F_1 \) and \( F_2 \).

**Theorem 6.** Assume \( A1–A5 \). Then, the ex-ante optimal SPBA gives the buyer higher expected surplus than any FPHA.

The proof relies on many of the same ideas as that of Theorem 5. As before, \( A \leq \tau \) do not make sense. We then show that for \( A > \tau \), \( \phi' > 1 \). Then, depending on whether \( \phi \) crosses \( \eta_M \), we either show a dominating second price auction as before, or we show that \( \phi \) is in fact dominated by \( \delta \). But then, by \( A5 \), we are again done.

Finally when costs are symmetric (but densities are not necessarily increasing), \( A5 \) is no longer needed and we can return to the stronger notion of dominance.

**Corollary 2.** Let \( F_1 = F_2 = F \) and \( \Delta > 0 \). Assume that \( F \) satisfies \( A2–A4 \). Then the optimal FPHA is dominated by an appropriately chosen SPBA.

**8. On the relationship to Maskin and Riley**

Consider an auction with one seller and two buyers, where the buyers have distributions \( G_1 \) and \( G_2 \) over values, where \( G_2 \) has support \([0, 1]\), and where for some \( t > 0 \), \( G_1 \) has support \([t, 1 + t]\) and has \( G_1(v) = G_2(v - t) \) for all \( v \). Assume also that \( g_1 \) is increasing. Maskin and Riley [36] show that a symmetric first price mechanism in this setting gives the seller higher expected revenue than a symmetric second price mechanism. Kirkegaard [21–23] extends this result to a larger class of settings and shows that the result is robust to the addition of a reserve price. Thus, there is a presumption for this setting that a first price mechanism is superior to a second. This presumption is premature.

**Proposition 6.** Let \( g_1 \) be increasing, and let \( \rho_{G_1} \) and \( \frac{\log g_1}{\log G_1} \) be minimized at \( t \). Then, any first price auction (including the one in which the players are treated symmetrically) is dominated by the optimal second price bonus auction.

The proof of this proposition involves first translating our results from a setting of a buyer and two sellers into a setting with a seller and two buyers (see Appendix B for details) and then recalling (per the discussion in Section 2) that an auction with shifted support is isomorphic to one with a handicap. In particular, begin by taking \( F_2(s) = G_2(1 - s) \) and \( F_1(s) = G_1(1 - s) \), with supports \([0, 1]\) and \([-t, 1 - t]\) respectively. Next, replace this by an FPHA auction in which \( F_1 \) is set equal to \( F_2 \), but \( A \) is increased by \( t \). Then, as Appendix B shows in detail, \( F_1 \) satisfies \( A2–A4 \) (the fact that \( f_1 \) is decreasing implies \( A2 \)). Corollary 2 thus applies to show that the optimal FPHA is dominated by the optimal SPBA and so a fortiori the optimal SPBA generates higher expected revenues that any particular FPHA, including in particular the one in which \( A \) is set equal to \( t \) (which corresponds to the symmetric auction for the shifted cost setting).

The key to understanding the relationship between the results is to note that both Maskin and Riley and Kirkegaard are comparing the particular first and second price auction in which the

\[47\] In particular, it can be shown that for any \( \varepsilon > 0 \), there is \( \Delta \) large enough that \( BS(\lambda_{t+\varepsilon}) > BS(\delta_t) \).
players are being treated symmetrically. While natural, this turns out to be a pretty bad choice for the auctioneer, especially in the second price case, where this format systematically distorts the allocation too far in favor of 1.

In contrast, we show that a well chosen asymmetric second price auction will perform better in the Maskin and Riley setting than does the symmetric first price auction. In fact, we show that the best the auctioneer can do by any first price auction, symmetric or otherwise, is dominated by an appropriate second price auction, and that in expected surplus terms, the optimal second price auction outperforms anything that can be achieved using a first price mechanism. So, while Maskin and Riley and Kirkegaard show that between a focal pair of asymmetric auctions one prefers the first price mechanism, we show that if one can choose which handicap to offer, one will prefer a second price mechanism.

9. Conclusion

Our results suggest an underexploited connection between $\rho$-concavity and the study of auctions and mechanism design more generally. That the slope of the bid function is one to one with the local $\rho$-concavity of the associated surplus function and that this in turn is tightly tied to the $\rho$-concavity properties of the underlying distributions certainly suggests that the connection is worth exploration. For example, it would be interesting to see what these tools have to say about first-price auctions with more than 2 players.

We show that for a class of auctions, second-price mechanisms are superior to first price mechanisms. These results should be interesting to an economic theorist, but also to firms that engage in procurement.

The derivation of the result that $\phi'(\cdot) > 1$ uses techniques that are new, and that seem likely have wider applicability. The degree to which one can generate bounds on the surplus available to each player, and use that to partially characterize equilibrium bid functions seems intriguing. An obvious topic for further research is to get a better understanding of the examples suggesting that $\phi'(\cdot) > 1$ holds more widely than under our conditions.

Our ranking results are primarily in terms of dominance. Further exploration using expected buyer surplus is merited, as is auction design when quality is affected by pre-auction effort.48 Other simple auction forms, such as percentage-bonus auctions, deserve more consideration, especially because of their wide-spread use in practice.49 The question of exclusion is also interesting. Extending our analysis to these environments is a question for further research.

Appendix A. Proofs

A.1. Proof of Theorem 1

That $\phi(0) = c_1$ follows since a bid of $\beta_2(0) + A$ by 1 wins with probability one and thus dominates any lower bid, and similarly for a bid of $\beta_1(c_1) - A$ by 2. Similarly, $\phi(1 - A) = \tilde{c}_1$.

49 Marion [34,35] looks at federal highway contracts with minority preferences which are governed by a proportional mechanism. In a recent paper, Kirkegaard [22] argues that the competition for Canadian government research grants is best described by a combination of proportional and fixed handicaps. See also related experimental results by Schotter and Weigelt [49].
If 1 with type $c$ bids as if his type is $\tilde{c}$, his surplus is $S_1(\tilde{c}; c) = \tilde{F}_2(\psi(\tilde{c}))(\beta_1(\tilde{c}) - c)$. By the envelope theorem,

$$S_1'(c) = \frac{\partial}{\partial c} S_1(\tilde{c}; c) \bigg|_{\tilde{c}=c} = -\tilde{F}_2(\psi(c)).$$

(10)

Given that $b_1$ is restricted to be at most $\tilde{c}_1$, $S_1(\tilde{c}_1) = 0$, yielding the expression for $S_1$. Similarly,

$$S_2'(c) = -\tilde{F}_1(\phi(c)),$$

(11)

and for $c_2 > \tilde{c}_1 - A$ no $b_2 > c_2$ ever wins, and so $S_2(\tilde{c}_1 - A) = 0$, yielding (1).

Since $\tilde{F}_2(\psi(c))(\beta_1(c) - c) = S_1(c)$, and by (10), $\beta_1(c) = c + \frac{S_1(c)}{-S_1'(c)}$. But then, wherever $\psi$ is differentiable,

$$\beta_1'(c) = 1 + \frac{S_1'(c)(-S_1'(c)) + S_1(c)S_1''(c)}{(S_1'(c))^2} = W_{S_1}(c),$$

and similarly for $\beta_2'$, giving (2).

As strictly increasing functions, $\beta_1$ and $\beta_2$ are differentiable almost everywhere. And, as $\beta_1(\phi(c)) = \beta_2(c) + A$, $\phi$ is continuous, and where $\beta_1'$ and $\beta_2'$ exist,

$$\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))} > 0.$$  

(12)

Substituting (2) into (12) gives

$$\phi'(c) = \frac{W_{S_2}(c)}{W_{S_1}(\phi(c))} = \frac{S_2(c)f_1(\phi(c))}{S_1(\phi(c))}\frac{\phi'(c)}{F_2(c)F_1'(\phi(c))}.$$  

(13)

using (10) and (11) to evaluate $W_S$ and $W_{S_1}$. Canceling $\phi'(c) > 0$ and rearranging yields (3).

Now, let us show that $\phi \in C^{k+1}[0, \tilde{c}_1 - A]$. We show the result on $[0, a]$, $a < \tilde{c}_1 - A$. Since $a$ is arbitrary, the result follows. We have that $\phi$ is $C^0[0, a]$. Assume that $\phi \in C^k[0, a]$ where $0 \leq k \leq k$. Then, $f_1(\phi)$ and $\tilde{F}_1(\phi)$ belong to $C^k[0, a]$. Since $S_2 = \int_{\tilde{c}_1 - A}^c \tilde{F}_1(\phi(s))\,ds$ it follows from the fundamental theorem of calculus that $S_2 \in C^{k+1}[0, a]$ and similarly for $S_1$. But, as a bounded, continuous function on a compact interval, $\phi$ is absolutely continuous and so

$$\phi(c) = \phi(0) + \int_0^c \phi'(t)\,dt = \xi_1 + \int_0^c \frac{S_1(\phi(t))}{S_2(t)}\frac{f_2(c)}{f_1(\phi(c))}\,dt.$$  

As each part of the integrand is positive and belongs to $C^k[0, a]$, it follows that $\phi \in C^{k+1}[0, a]$. By induction, $\phi \in C^{k+1}[0, a]$. It follows that $\beta_1 \in C^{k+1}[0, \tilde{c}_1)$ and $\beta_2 \in C^{k+1}[0, \tilde{c}_1 - A]$.

A.2. Proofs for Sections 3 and 4

We use a series of lemmas.

**Lemma 10.** If $f$ is a log-concave density then $F$ and $\tilde{F}$ are themselves log-concave. If $f$ is a (strictly) log-concave density then $\frac{f}{F}$ is (strictly) decreasing and $\frac{f}{\tilde{F}}$ is (strictly) increasing.
For a proof see, e.g., Prekopa [42].

**Lemma 11.** Let $g$ be positive and log-concave on some interval near 1, with $g(1) = 0$ and with $W_g(1)$ and $W_G(1)$ well defined in the extended real line. Then,

$$W_G(1) = \frac{1}{2 - W_g(1)}. \quad (14)$$

**Proof.** Let $c^*$ be such that $g$ is log-concave on $[c^*, 1]$ and $g'(c^*) < 0$. By log-concavity, $g'(s) < 0$ for all $s \in [c^*, 1)$. Thus, l’Hôpital’s rule applies to give

$$W_G(1) = \lim_{s \to 1} \left( -\frac{g'(s)\tilde{G}(s)}{g^2(s)} \right) = -\lim_{s \to 1} \frac{g''(s)\tilde{G}(s) - g'(s)g(s)}{2g'(s)g(s)}$$

$$= -\lim_{s \to 1} \frac{g''(s)\tilde{G}(s)}{2g'(s)g(s)} + \frac{1}{2} = \frac{1}{2} \lim_{s \to 1} \left( -\frac{g''(s)g(s)g'(s)\tilde{G}(s)}{(g')^2(s)g^2(s)} \right) + \frac{1}{2}$$

$$= \frac{1}{2} \lim_{s \to 1} (W_G(s)W_g(s)) + \frac{1}{2}. \quad (15)$$

Now, since $g$ is log-concave $W_g(1) \in (-\infty, 1]$. By Lemma 10 $\tilde{G}$ is log-concave on $[c^*, 1]$ and so, since $g$ is decreasing near zero, $W_G(1) \geq 0$, and so $W_G(1) \leq 1$ is finite. If $W_g(1) = -\infty$, then for (15) to be satisfied, we must have $W_G(1) = 0$, and so (14) is satisfied. If $W_g(1)$ is finite, then

$$W_G(1) = \frac{1}{2} \lim_{s \to 1} (W_G(s)W_g(s)) + \frac{1}{2} = \frac{1}{2} W_G(1)W_g(1) + \frac{1}{2}$$

and rearranging yields the result. \[ \Box \]

The following lemma shows why the assumption that $W_g(1)$ is finite is mild.

**Lemma 12.** Assume that $g$ is $C^\infty[0, 1]$ and has $g(1) = 0$ and $g^{(k)}(1) < \infty$ for all $k$. Then, $W_g(1) \in (0, 1)$. In particular, if we let $n$ be such that $g^{(n)}(1) \neq 0$ while $g^{(k)}(1) = 0$ for all $k \in \{0, 1, \ldots, n-1\}$, then $W_g(1) = \frac{n}{n+1}$. \[ 50 \]

**Proof.** Since $g^{(n)}(1) \neq 0$ while $g^{(n-1)}(1) = 0$,

$$W_{g^{(n-1)}}(1) = \frac{g^{(n-1)}(1)g^{(n+1)}(1)}{(g^{(n)}(1))^2} = 0.$$ 

Assume by induction that $W_{g^{(n-k)}}(1) = \frac{k-1}{k}$ for some $k \in \{1, 2, \ldots, n\}$. Then, since $g^{n-k}(1) = 0$, Lemma 11 applies to $g^{(n-k)}$ (which, since $W_{g^{(n-k)}}(1) < 1$, is log-concave on some interval near 1) to yield

$$W_{g^{(n-(k+1))}}(1) = \frac{1}{2 - W_{g^{(n-k)}}(1)}$$

$$= \frac{k}{k + 1}. \quad \Box$$

50 Note that $n$ must be finite, otherwise by Taylor’s expansion $f \equiv 0$. 
Proof of Theorem 2. We will prove (4), the other case is similar. Let \( \tau = \min[0,c] \rho g(s) \), and let \( \alpha = \frac{g'\tau(c)}{g'\tau(c)} = \frac{1}{1 + \frac{\rho g(c)}{1 + \rho g(c)}} \). Since \( g'\tau \) is concave on \([0,c]\), and since \( (g'\tau)'(c) > 0 \), we have that \( \alpha \geq c \), and that for all \( s \in [0,c] \)

\[
g'\tau(s) \leq g'(c) + (g'\tau)'(c)(s - c) = g'(c) + \frac{1}{\alpha} (\alpha + s - c),
\]

and so

\[
g(s) \leq g(c)\alpha^{-\frac{1}{\tau}}(\alpha + s - c)^{\frac{1}{\tau}}.
\]

Hence,

\[
G(c) = \int_0^c g(s) \, ds \\
\leq g(c)\alpha^{-\frac{1}{\tau}} \int_0^c (\alpha + s - c)^{\frac{1}{\tau}} \, ds \\
\leq g(c)\alpha^{-\frac{1}{\tau}} \int_{c-\alpha}^c (\alpha + s - c)^{\frac{1}{\tau}} \, ds \\
= g(c)\alpha^{-\frac{1}{\tau}} \int_{c-\alpha}^c \frac{1}{\tau + 1} (\alpha + s - c)^{\frac{1}{\tau} + 1} \, ds \\
= g(c) \frac{\tau}{\tau + 1} \alpha.
\]

But, \( \alpha = \frac{g'\tau(c)}{g'\tau(c)} = \frac{1}{1 + \frac{\rho g(c)}{1 + \rho g(c)}} \). Substituting and rearranging, we have (using \( g'(c) > 0 \)) that

\[
W_G(c) \leq \frac{1}{1 + \tau}
\]
or

\[
\rho_G(c) \geq \frac{\rho g(c)}{1 + \rho g(c)}
\]

Similarly, let \( \tau = \max[0,c] \rho g(s) \) so that \( g'\tau \) is convex on \([0,c]\). Then, \( \alpha < c \), and

\[
g(s) \geq g(c)\alpha^{-\frac{1}{\tau}}(\alpha + (s - c))^{\frac{1}{\tau}}
\]

and hence

\[
\int_0^c g(s) \, ds \geq \int_{c-\alpha}^c g(s) \, ds \geq g(c)\alpha^{-\frac{1}{\tau}} \int_{c-\alpha}^c (\alpha + (s - c))^{\frac{1}{\tau}} \, ds
\]

from which

\[
\frac{\rho g(c)}{1 + \rho g(c)} \geq \rho_G(c).
\]

Proof of Lemma 3. Assume that \( g(0) = 0 \), and that \( \rho g \) is decreasing on \([0,\hat{c}]\), so that \( W_g \) is increasing. Note that
\[
W_G'(c) = W_G(c) \left( \frac{g(c)}{G(c)} - 2 \frac{g'(c)}{g(c)} + \frac{g''(c)}{g'(c)} \right)
\]
\[
= W_G(c) \left( \frac{g'(c)}{g(c)} \right) \left( \frac{1}{W_G(c)} - 2 + W_g(c) \right)
\]

where
\[
A(c) = G(c) \left( \frac{g'(c)}{g(c)} \right)^2 > 0.
\]

But then, where \(W_G'(c) = 0\), \(B(c) = 0\), and so \(W_G''(c) = A(c)W_g'(c) \geq 0\), and so on \([0, \hat{c}]\), \(W_G'(c)\) single crosses 0 from below (or does not cross at all).

Assume that for some \(\tilde{c} \in [0, \hat{c}]\), \(W_G'(\tilde{c}) < 0\). Then, by single crossing \(W_G'(c) < 0\) for all \(c < \tilde{c}\), and so \(B(c)\) is increasing everywhere on \([0, \tilde{c}]\). This is a contradiction, because by Lemma 11, \(B(0) = 0\), while since \(W_G'(\tilde{c}) < 0\), \(B(\tilde{c}) < 0\). Hence \(W_G\) is everywhere weakly positive on \([0, \tilde{c}]\), and thus, \(\rho_G\) is decreasing on \([0, \hat{c}]\). The other three cases are similar. \(\square\)

**Remark 1.** Let \(g\) be log-concave. If \(g\) is increasing at any given \(c\), then \(g'\) is positive and decreasing at \(c\), as is \(\bar{G}\) (because \(\bar{G}\) is also log-concave). Thus, \(W_{\bar{G}}' = -\left(\frac{g'\bar{G}}{g^2}\right)\) is increasing at \(c\). Similarly, if \(g\) is decreasing at any given \(c\), then \(W_G\) is decreasing at \(c\).

**Proof of Lemma 1.** Assume \(W_{\bar{F}}\) is increasing and \(\gamma\) is concave. Then,
\[
W_{\bar{F}\gamma}(c) = \frac{\gamma'(c) f(\gamma(c)) \int_c^1 \bar{F}(\gamma(s)) \, ds}{(\bar{F}(\gamma(c)))^2}
\]
and so if
\[
\gamma'(c) \int_c^1 \bar{F}(\gamma(s)) \, ds \geq \int_{\gamma(c)}^1 \bar{F}(s) \, ds
\]
then
\[
W_{\bar{F}\gamma}(c) \geq W_{\bar{F}}(\gamma(c)) \geq W_{\bar{F}}(c)
\]
since \(W_{\bar{F}}\) is increasing and \(\gamma(c) \geq c\).

To prove (16) define
\[
Q(c) = \gamma'(c) \int_c^1 \bar{F}(\gamma(s)) \, ds - \int_{\gamma(c)}^1 \bar{F}(s) \, ds.
\]
Since \(\gamma\) is concave, \(\lim_{c \to 1} \gamma'(c) < \infty\), and so \(Q(1) = 0\). Also,
\[
Q'(c) = \gamma''(c) \int_c^1 \bar{F}(\gamma(s)) \, ds - \gamma'(c) \bar{F}(\gamma(c)) + \gamma'(c) \bar{F}(\gamma(c))
\]
\[
= \gamma''(c) \int_c^1 \bar{F}(\gamma(s)) \, ds \leq 0
\]
again since $\gamma$ is concave. Since $Q' \leq 0$ and $Q(1) = 0$, $Q(c) \geq 0$ or equivalently (16). The other case is similar. □

Proof of Lemma 2. Define
\[
Q(c) = \alpha \int_c^1 g^\alpha(s)ds - g^{\alpha - 1}(c) \int_c^1 g(s)ds.
\]
Note that $Q(1) = 0$ and that
\[
Q'(c) = -\alpha g^\alpha(c) + g^\alpha(c) - (\alpha - 1)g'(c)g^{\alpha - 2}(c) \int_c^1 g(s)ds
\]
\[
= (1 - \alpha)g^\alpha(c) \left(1 + \frac{g'(c) \int_c^1 g(s)ds}{g^2(c)}\right)
\]
\[
= (1 - \alpha)g^\alpha(c) \rho_{f_g}(c) \leq 0
\]
since by assumption $\rho_g \geq 0$ and so by Theorem 2 $\rho_{f_g}(c) \geq 0$. Thus $Q(c) \geq 0$.

Proof of Theorem 3. From Theorem 1
\[
\phi'(r) = \frac{S_1(\phi(r)) f_2(r)}{S_2(r) f_2(\phi(r))} \equiv \frac{T}{B},
\]
and so $\frac{\phi''}{\phi'} = (\log \phi')'$ is
\[
\phi'(r) \frac{S'_1}{S_1}(\phi(r)) - \frac{S'_2}{S_2}(r) + \left(\log \frac{f_2}{F_2^2}(r)\right)'(r) - \phi'(r) \left(\log \frac{f_1}{F_1^2}\right)'(\phi(r)).
\]
By log-concavity of $\bar{F}_2$, $(\log \frac{f_2}{F_2^2})' > 0$, and so
\[
\left(\log \frac{f_2}{F_2^2}\right)'(r) \geq \phi'(r) \left(\log \frac{f_2}{F_2^2}\right)'(r).
\]
Note that $(\log \frac{f_1}{F_1^2})'(r) = \frac{f'_1}{f_1^2}(r) - 2 \frac{f'_2}{F_2^2}(r)$ and similarly for $(\log \frac{f_2}{F_2^2})'$. From (1), $S'_1(\phi(r)) = -\bar{F}_2(r)$, and so
\[
\frac{S'_1}{S_1}(\phi(r)) = -\frac{\bar{F}_2(r)}{S_1(\phi(r))} = -\frac{\bar{F}_2(r)}{T}
\]
and similarly
\[
\frac{S_2'(r)}{S_2(r)} = -\frac{\bar{F}_1'(\phi(r))}{\bar{F}_1(\phi(r))} = -\phi'(r) \frac{f_1(\phi(r))}{T}.
\]  
(21)

Substitute (19), (20), and (21) into (18), collect terms, and cancel \(\phi'(r) > 0\) to obtain

\[
\phi''(r) \geq s \left( \frac{1}{T} - 2 \right) \left( \frac{f_1(\phi(r))}{\bar{F}_1(\phi(r))} - \frac{f_2(\phi(r))}{\bar{F}_2(\phi(r))} \right) + \left( \frac{f_2'(r)}{f_2(r)} - \frac{f_1'(\phi(r))}{f_1(\phi(r))} \right),
\]

where \(\geq\) means that if the right hand side is weakly positive, then so is the left hand side, and that if the left hand side is strictly positive, then so is the right hand side.

Now,

\[
S_2(r) = \int_r^{\tilde{c}_1-A} \bar{F}_1(\phi(s)) \, ds = \int_{\phi(r)}^{\phi(\tilde{c}_1-A)} \bar{F}_1(s) \psi'(s) \, ds
\]

\[
< \frac{1}{\phi'(r)} \int_{\phi(r)}^{\tilde{c}_1} \bar{F}_1(s) \, ds,
\]

where the strict inequality follows since \(r\) is a global minimum of \(\phi'\), with \(\phi'(r) \leq 1\), since \(\phi'\) is continuous, and since \(\lim inf_{c \to \tilde{c}_1-A} \phi'(c) > 1\). Multiplying both sides by \(\phi'(r) f_1(\phi(r))\) yields

\[
\phi'(r)B < W \bar{F}_1(\phi(r)).
\]

Since \(T = \phi'(r)B\), we have

\[
T < W \bar{F}_1(\phi(r)).
\]

(23)

Substituting (23) into (22) yields the result. □

**Proof of Proposition 5.** By Lemma 4, if \(\phi'(c_2) \leq 1\) anywhere, then \(\phi'\) has an interior global minimum \(r\) with \(\phi'(r) \leq 1\). Since \(f\) is increasing, \(\bar{F}\) is concave, and so by Theorem 2, \(\frac{1}{\bar{F}(\phi(r))} - 2 \geq 0\), while by Lemma 4 and log-concavity of \(f\) and \(\bar{F}\), \(\frac{f}{\bar{F}}(\phi(r)) - \frac{f}{\bar{F}}(r)\) and \(\frac{f}{\bar{F}}(r) - \frac{f}{\bar{F}}(\phi(r))\) are each non-negative. But then, \(H(r)\) is non-negative, and so by Theorem 3, \(\phi''(r) \geq 0\), contradicting that \(\phi'\) is minimized at \(r\). □

**Proof of Lemma 5.** Since \(\phi'(c) \geq 1\), and \(\bar{F}\) is log-concave,

\[
\left( \ln \frac{\bar{F}(\phi(c))}{\bar{F}(c)} \right)' = \phi'(c) (\ln \bar{F})'(\phi(c)) - (\ln \bar{F})'(c) \\
\leq (\ln \bar{F})'(\phi(c)) - (\ln \bar{F})'(c) \leq 0
\]

using \(\phi(c) \geq c\). Thus, \(\frac{\bar{F}(\phi(c))}{\bar{F}(c)}\) is decreasing, and so

\[
\frac{\int_c^{\tilde{c}_1-A} \bar{F}(\phi(s)) \, ds}{\int_c^1 \bar{F}(s) \, ds} \leq \frac{\bar{F}(\phi(c))}{\bar{F}(c)}.
\]

But then, \(\beta_s(c) = c + \int_c^1 \bar{F}(s) \, ds \geq c + \frac{\int_c^{\tilde{c}_1-A} \bar{F}(\phi(s)) \, ds}{\bar{F}(\phi(c))} = \beta_2(c)\) and analogously, \(\beta_1(c) \geq \beta_s(c)\). □

**Proof of Lemma 6.** Adapting Myerson [40] or Riley and Samuelson [46] in the obvious manner, the buyer’s expected surplus from mechanism \(\eta\) is
\[ BS(\eta) = v_2 - 1 + \int_{c_1 < \eta(c_2)} \left( \Delta - \omega_1(c_1) + \omega_2(c) \right) f_1(c_1) f_2(c_2) dc_1 dc_2. \]  

This is intuitive. Always buying from 2 gives the buyer surplus \( v_2 - 1 \), since 2 must receive 1 if he is to sell for all \( c_2 \). The integral represents the change in buyer surplus from buying from 1 according to \( \eta \). Thus, among mechanisms that always buy, 1 optimally wins if \( \Delta > \omega_1(c_1) - \omega_2(c_2) \) and 2 wins otherwise. From (24), it also follows that if one mechanism dominates another, then it also gives the buyer higher surplus.

A.3. Proof of Theorem 5

We will prove Theorem 5 (and 6) closely following the geometric arguments behind our previous ranking result, establishing that \( \eta'_M < 1 \) in the relevant domain, and, using Theorem 3, that \( \phi' > 1 \). Most of the development in this section is also needed for the proof of Theorem 6. So, in what follows, we will at many places replace the assumption that \( f \) is increasing by \( A3 \) (which is weaker by Corollary 3).

A.3.1. Geometry of \( \gamma \) and \( \eta_M \)

We will start with some geometric properties that link \( \gamma \) to \( \eta_M \). Define \( \gamma \) by

\[ f_1(\gamma(\gamma(c))) = f_2(c). \]

Lemma 13. Under \( A1 \) and \( A3 \), \( \gamma \) \( \geq \gamma \) and \( (\gamma)^' \geq 1 \).

Proof. Differentiating \( \tilde{F}_1(\gamma(c)) = \tilde{F}_2(c) \) we have \( f_1(\gamma(c))\gamma'(c) = f_2(c) \), and so

\[ \frac{f_2}{\tilde{F}_2}(c) = \gamma'(c) \frac{f_1}{\tilde{F}_1}(\gamma(c)). \]

Thus, since \( \gamma' \geq 1 \),

\[ \frac{f_1}{\tilde{F}_1}(\gamma(c)) - \frac{f_2}{\tilde{F}_2}(c) = (1 - \gamma'(c)) \frac{f_1}{\tilde{F}_1}(\gamma(c)) \leq 0. \]

As \( \frac{f_1}{\tilde{F}_1} \) is increasing, it follows that \( \gamma_\ell \geq \gamma \), and in particular, \( \gamma_\ell(0) \geq \gamma(0) \).

Also since

\[ f'_1(\gamma(c))(\gamma'(c))^2 + f_1(\gamma(c))\gamma''(c) = f'_2(c), \]

we have

\[ W_{\tilde{F}_2}(c) = \frac{\tilde{F}_1(\gamma(c))(-f'_1(\gamma(c))\gamma'(c)(\gamma'(c))^2 - f_1(\gamma(c))\gamma''(c))}{(f_1(\gamma(c))\gamma'(c))^2} \]

\[ = W_{\tilde{F}_1}(\gamma(c)) - \frac{\tilde{F}_1(\gamma(c))}{\tilde{F}_1(\gamma(c))} \frac{\gamma''(c)}{(\gamma'(c))^2} \]

\[ \leq W_{\tilde{F}_1}(\gamma(c)), \]

(25)

As in footnote 16, since the objects are monotone, there is no ambiguity in defining \( \gamma_\ell(c_2) = \tilde{c}_1 \) if \( \frac{f_1}{\tilde{F}_1}(c_1) < \frac{f_2}{\tilde{F}_2}(c_2) \) for all \( c_1 \), and analogously if the inequality is reversed.
since $\gamma'' \geq 0$. But then, since $W_{F_1}$ is increasing, and since $\gamma'_F(c) \geq \gamma(c)$,

$$W_{F_1}(\gamma'_F(c)) \geq W_{F_2}(c).$$

Finally, differentiating $\frac{\dot{F}_1}{F_1}(\gamma'_F(c_2)) = \frac{\dot{F}_2}{F_2}(c_2)$, noting that $W_{F_1} \leq 1$ by log-concavity, and using (26) we have

$$\gamma'_F(c) = \frac{-1 + W_{F_2}(c)}{-1 + W_{F_1}(\gamma'_F(c))} \geq 1. \quad \square$$

Lemma 14. Under A1 and A2, $\eta_M$ crosses $\gamma$ at most once on $[0, 1)$, and, if it crosses, does so from above. Anywhere that $\eta_M(c) \geq \gamma(c)$, $\eta'_M(c) \leq 1$.

Proof. Differentiating (7) we have

$$\eta'_M(c_2) = \frac{1 + \rho_{F_2}(c_2)}{1 + \rho_{F_1}(\eta_M(c_2))}.$$ (27)

By computations analogous to (25) we have

$$W_{F_1}(\gamma(c)) + \left(\frac{F_1(\gamma(c))}{f_1(\gamma(c))}\right) \frac{\gamma''(c_2)}{(\gamma'(c_2))^2} = W_{F_2}(c_2)$$

and so $\rho_{F_1}(\gamma(c_2)) \geq \rho_{F_2}(c_2)$. So, since $\rho_{F_1}$ is increasing by A2, for any $c_1 \geq \gamma(c_2)$, $\rho_{F_1}(c_1) \geq \rho_{F_2}(c_2)$. In particular at a point where $\eta_M(c_2) \geq \gamma(c_2)$ we thus have by (27) that

$$\eta'_M(c_2) \leq 1 \leq \gamma'(c_2). \quad \square$$

A.3.2. Geometry of $\gamma$ and $\phi$

Now, we examine the relationship of $\gamma$ and $\phi$.

Lemma 15. Anywhere that $\phi(c_2) \leq \min(\gamma'_F(c_2), c_2 + A)$, $\phi'(c_2) \geq 1$. Anywhere that $\phi(c_2) \geq \max(\gamma'_F(c_2), c_2 + A)$, $\phi'(c_2) \leq 1$. Each inequality is strict unless $\phi(c_2) = \gamma'_F(c_2) = c_2 + A$.

Proof. From Theorem 1, for any $c_2 > 0$ such that $c_1 = \phi(c_2)$ we have

$$\phi'(c_2) = \frac{S_1(c_1) f_2(c_2)}{F_2(c_2)} - \frac{S_2(c_2) f_1(c_1)}{F_1(c_1)}$$

$$= \frac{\beta_1(c_1) - c_1}{\beta_2(c_2) - c_2} \frac{f_2(c_2)}{F_2(c_2)} - \frac{f_1(c_1)}{F_1(c_1)}$$

$$= \left(1 + \frac{A - (c_1 - c_2)}{\beta_2(c_2) - c_2}\right) \frac{f_2(c_2)}{F_2(c_2)} - \frac{f_1(c_1)}{F_1(c_1)},$$ (28)

since $\beta_1(c_1) = \beta_2(c_2) + A$ by the definition of $\phi$. When $\phi(c_2) \leq \min(\gamma'_F(c_2), c_2 + A)$, each term in (28) is at least 1, while when $\phi(c_2) \geq \max(\gamma'_F(c_2), c_2 + A)$, each term is at most 1. \quad \square
For the case $f$ increasing, define $c_M$ as the first point at which $\eta_M(c_2) = \bar{c}_1$. Since $\Delta - \omega(\bar{c}_1) + \omega(1) > 0$, then $c_M < 1$.

**Lemma 16.** Under the conditions of Theorem 5, $\eta_M \geq \lambda_{\bar{c}_1 - c_M}$.

This follows since $\eta_M(c_M) = \bar{c}_1 > \gamma(c)$ (since $c_M < 1$) and thus from Lemma 14, $\eta_M$ is everywhere above $\gamma$ and has $\eta'_M \leq 1$. On the other hand $\eta_M(c_M) = \lambda_{\bar{c}_1 - c_M}(c_M)$ and $\lambda'_{\bar{c}_1 - c_M} = 1$, and so $\eta_M \geq \lambda_{\bar{c}_1 - c_M}$ before $c_M$.

These results in hand, we can begin by showing that first price mechanisms with $A \leq \tau$ do not make sense.

**Lemma 17.** Under the conditions of Theorem 5 if $A \leq \tau$, then $\text{FPHA}_A$ is dominated by $\text{SPBA}_{\bar{c}_1 - c_M}$.

**Proof.** Since $\bar{c}_1 - c_M > \bar{c}_1 - 1 = \tau$, and by Lemma 16, we have $\eta_M \geq \lambda_{\bar{c}_1 - c_M} > \lambda_{\tau}$. It is thus enough to show that since $\phi$ starts below $\lambda_{\tau}$, it cannot get above it. But, from Lemma 13 $\phi(0) = \gamma(0) \leq \gamma_f(0)$. So, since $\gamma'_f(c_2) \geq 1$ and $\lambda'_A(c_2) = 1$ for all $c_2$, if $\phi(c_2) > \max(\gamma_f(c_2), \lambda_A(c_2))$ anywhere, then there is a $\hat{c}_2$ where $\phi'(\hat{c}_2) > 1$, and where $\phi(\hat{c}_2) \geq \max(\gamma_f(\hat{c}_2), \lambda_A(\hat{c}_2))$. This contradicts Lemma 15. So, $\phi \leq \max(\gamma_f, \lambda_A) < \lambda_{\tau}$. \qed

### A.3.3. Boundary conditions for $\phi'$

Given Lemma 17, we can restrict attention to $A > \tau$. For such $A$, we aim to show that $\phi' > 1$ everywhere on $(0, \bar{c}_1 - A)$. In this section, we show that this is true at the two boundaries of the domain of $\phi'$, i.e., at 0 and at $\bar{c}_1 - A$.

**Lemma 18.** Under A1 and A3, if $A > \tau$ then $\phi'(0) > 1$, and $\lim_{c \to \bar{c}_1 - A} \phi'(c) = \infty$.

So, $\phi'$ becomes arbitrarily large as $c \to \bar{c}_1 - A$. The key driver is that since $\bar{c}_1 - A < 1$, the behavior of $S_1$ at $\bar{c}_1$ and $S_2$ at $\bar{c}_1 - A$ are very different, with the surplus of player 1 changing much more quickly. Unlike much of the development to date, this result depends crucially on $A > \tau$.

**Proof of Lemma 18.** Note first that since $\bar{c}_1 - 1 = \tau, \bar{c}_1 - A < 1$. Since $\gamma' \geq 1, \gamma(1) - \gamma(0) \geq 1$, and so $\bar{c}_1 - 1 \geq \gamma(0)$, or

$A > \bar{c}_1 - 1 \geq \gamma(0) = \xi_1 - 0$.

Lemma 15 thus applies at 0 since $\phi(0) = \gamma(0) \leq \gamma_f(0)$ by Lemma 13 and thus $\phi'(0) > 1$.

Next, let us show that $\beta'_1(\bar{c}_1) = \beta'_2(\bar{c}_1 - A) = 0$. Note first that for any $c < \bar{c}_1, \beta_1(c)$ earns $\bar{F}_2(\phi(c))(\beta_1(c) - c)$, while a bid of $\bar{c}_1$ earns at least $\bar{F}_2(\bar{c}_1 - A)(\bar{c}_1 - c)$, and so

$\frac{\bar{F}_2(\phi(c))(\beta_1(c) - c) \geq \bar{F}_2(\bar{c}_1 - A)(\bar{c}_1 - c),}$

from which

$\frac{\beta_1(c) - c \geq \frac{\bar{F}_2(\bar{c}_1 - A)}{\bar{F}_2(\phi(c))},}$
and so, since $\psi(\tilde{c}_1) = \tilde{c}_1 - A$,

$$\lim_{c \to \tilde{c}_1} \inf \frac{\beta_1(c) - c}{\tilde{c}_1 - c} \geq 1. \quad (29)$$

Since $\beta_1$ is increasing, we have

$$\lim_{c \to \tilde{c}_1} \inf \frac{\beta_1(\tilde{c}_1) - \beta_1(c)}{\tilde{c}_1 - c} \geq 0. \quad (30)$$

But,

$$\frac{\beta_1(c) - c}{\tilde{c}_1 - c} + \frac{\beta_1(\tilde{c}_1) - \beta_1(c)}{\tilde{c}_1 - c} = \frac{\beta_1(\tilde{c}_1) - c}{\tilde{c}_1 - c},$$

and so, since $\beta_1(\tilde{c}_1) = \tilde{c}_1$,

$$\lim_{c \to \tilde{c}_1} \frac{\beta_1(c) - c}{\tilde{c}_1 - c} + \frac{\beta_1(\tilde{c}_1) - \beta_1(c)}{\tilde{c}_1 - c} = 1.$$

It follows that

$$\lim_{c \to \tilde{c}_1} \frac{\beta_1(\tilde{c}_1) - \beta_1(c)}{\tilde{c}_1 - c} = 0$$

or $\beta'_1(\tilde{c}_1) = 0$.

The first order condition for player 1’s profit at $c < \tilde{c}_1$ is

$$\bar{F}_2(\psi(c))\beta'_1(c) = f_2(\psi(c))\psi'(c)(\beta_1(c) - c).$$

But, $\psi'(c) = \frac{\beta'_1(c)}{\beta'_2(\psi(c))}$, and so

$$\bar{F}_2(\psi(c))\beta'_1(c) = f_2(\psi(c))\frac{\beta'_1(c)}{\beta'_2(\psi(c))}(\beta_1(c) - c).$$

Canceling $\beta'_1(c) > 0$, and rearranging,

$$0 < \beta'_2(\psi(c)) = \frac{f_2(\psi(c))}{\bar{F}_2(\psi(c))}(\beta_1(c) - c) < \frac{f_2(\tilde{c}_1 - A)}{\bar{F}_2(\tilde{c}_1 - A)}(\tilde{c}_1 - c).$$

Thus, as $c \to \tilde{c}_1$, $\beta'_2(\psi(c)) \to 0$, i.e., $\beta'_2(\tilde{c}_1 - A) = 0$.

Assume that along some sequence $c_k \to \tilde{c}_1 - A$, $\lim_{k \to \infty} \phi'(c_k) = \alpha \in (0, \infty)$. Note that

$$\lim_{k \to \infty} \frac{\beta_1(\phi(c_k)) - \phi(c_k)}{\beta_2(c_k) - c_k} = \lim_{k \to \infty} \frac{\beta'_1(\phi(c_k)) - 1}{\beta'_2(c_k) - 1} \phi'(c_k) = \lim_{k \to \infty} \phi'(c_k). \quad (31)$$

by l’Hôpital’s rule and the previous step. But, by Theorem 1

$$\phi'(c) = \frac{\beta_1(\phi(c)) - \phi(c)}{\beta_2(c) - c} \frac{f_2(c)}{f_2(\phi(c))},$$

and so by (31), we have

$$\alpha = \alpha \lim_{k \to \infty} \frac{f_2(c_k)}{f_2(\phi(c_k))} = 0,$$

since $\phi(c) \to \tilde{c}_1$, while $\tilde{c}_1 - A < \tilde{c}_2$, a contradiction.
Since $\phi'$ is continuous, weakly positive, and has no cluster point in $(0, \infty)$, it follows that $\lim_{c \to \bar{c}_1 - A} \phi'(c)$ exists and is in $\{0, \infty\}$. So, assume that $\lim_{c \to \bar{c}_1 - A} \phi'(c) = 0$. Then, for any small $t$, there is a last $c(t)$ at which $\phi'(c) = t$ (this is well defined since $\phi$ is continuously differentiable and $[0, \bar{c}_1 - A]$ is compact). But, by a change of variables,

$$S_1(\phi(c(t))) = \int_{\phi(c(t))}^{\bar{c}_1} \bar{F}_2(\psi(s)) \, ds$$

$$= \int_{c(t)}^{\bar{c}_1 - A} \bar{F}_2(s) \phi'(s) \, ds$$

$$< t(\bar{c}_1 - A - c(t)),$$

since $\phi'(s) < t$, and $\bar{F}_2 < 1$ for $s > c(t)$.

Thus, since $\frac{f_2}{\bar{F}_2}$ is increasing

$$S_1(\phi(c(t))) \frac{f_2}{\bar{F}_2}(c(t)) < t(\bar{c}_1 - A - c(t)) \frac{f_2}{\bar{F}_2}(\bar{c}_1 - A).$$

The RHS converges to 0 as $t \to 0$ and $c(t) \to \bar{c}_1 - A$. But then the term $\frac{1}{t} - 2$ in (22) diverges for $r = c(t)$ for small $t$, the term $\frac{f_1(\phi(c(t)))}{\bar{F}_1(\phi(c(t)))} - \frac{f_2(c(t))}{\bar{F}_2(c(t))}$ diverges as well (noting that $c(t) < \bar{c}_1 - A < \bar{c}_2$), and, by log-concavity of $\tilde{f}$, the remaining term does not go to $-\infty$. Hence, $\phi''(c(t)) > 0$, contradicting the construction of $c(t)$.

We thus have that $\lim_{c \to \bar{c}_1 - A} \phi'(c) = \infty$. \(\square\)

A.3.4. Interior minima and the completion of the proof of Theorem 5

Given Lemma 18, if $\phi' \leq 1$ anywhere on $[0, c_1 - A]$, then $\phi'$ achieves a global minimum at some $r \in (0, c_1 - A)$ with $\phi'(r) \leq 1$. Given that $\phi$ is $C^2$, and that $r$ is interior, $\phi''(r) = 0$. Observe first that $\phi(r) \leq r + A$. To see this, note that since $\lambda_A$ lies weakly above $\gamma$ (recall that $\gamma \leq \lambda_2$ and $A > \tau$) it follows from Lemma 15 that if $\phi(c_2) \geq \lambda_A(c_2)$, then $\phi'(c_2) \leq 1$. As $\lambda_A = 1$, and since $\phi(0) = \gamma(0) \leq \lambda_A(0)$, $\phi$ cannot get above $\lambda_A$. But, from (28), since by the previous observation, $1 + \frac{A - \phi(r) - \tau}{\beta(r) - \tau} \geq 1$, and so, as $\phi'(r) \leq 1$, it must be that $\frac{f_2}{\bar{F}_2}(r) \leq \frac{1}{\bar{F}_1}(\phi(r))$.

But then, all requirements of Theorem 3 are met and so, since $f_1$ is increasing, we reach a contradiction as in Proposition 5. Hence, $\phi' > 1$ everywhere, and so, since $\eta'_M \leq 1$ by A2, an SPBA defined by their unique crossing dominates $\phi$.

A.4. Proofs for Section 7

Proof of Lemma 8. Note that

$$f_1 = -\tilde{F}_1 = n\tilde{G}^{n-1}g,$$

and so

$$\frac{f_1}{\bar{F}_1} = n\frac{g}{\bar{G}} \quad \text{and} \quad \frac{f_1'}{f_1} = -(n - 1)\frac{g}{\bar{G}} + \frac{g'}{g}. \quad (32)$$

\(52\) At the point that (22) is derived, we have used only that $\phi'(r) < 1$, which holds for $r = c(t)$ for small $t$ by definition, and none of the other properties assumed in the statement of Theorem 3.
Thus,
\[
W_{\tilde{F}_1} = \frac{\tilde{F}_1 - f_1'}{f_1} = \frac{1}{n} \frac{\tilde{G}}{g} \left( (n - 1) \frac{g}{\bar{G}} - g' \right)
\]
\[
= \frac{n - 1}{n} + \frac{1}{n} W_{\tilde{G}}.
\]
So, \(W_{\tilde{F}_1}\) has the same monotonicity as \(W_{\tilde{G}}\), and thus they will be maximized (and minimized at the same points).

Using (32), note that
\[
\frac{(\log f_1)''}{(\log \tilde{F}_1)''} = -\frac{(f_1')'}{(f_1')'} = \frac{(n - 1)(\frac{g}{\bar{G}})' - (\frac{g'}{g})'}{n(\frac{g}{\bar{G}})'}
\]
\[
= \frac{(n - 1)}{n} + \frac{1}{n} \frac{(\log g)''}{(\log \bar{G})''}.
\]
So, \(\frac{(\log f_1)''}{(\log \tilde{F}_1)''}\) and \(\frac{(\log g)''}{(\log \bar{G})''}\) reach their minima at the same point. ✷

**Proof of Lemma 9.** By definition \(\Delta + \omega_2(c) = \omega_1(\eta_M(c))\). Since \(f_1(\tilde{c}) = f_2(1) = 0\) and since \(\omega_1(c) = c + \frac{F_1}{f_1}(c)\), the left-hand side diverges as \(c_2\) approaches 1 and thus \(\eta_M(c) \to \tilde{c}_1\). Imagine there exists \(\hat{c} \in (0, 1)\) such that \(\eta_M(c) > \gamma(c)\) for all \(c \in (\hat{c}, 1)\). Thus for all \(c \in (\hat{c}, 1)\)
\[
\Delta > \omega_1(\gamma(c)) - \omega_2(c)
\]
and since \(F_1(\gamma(c)) = F_2(c)\) and \(\gamma'(c) f_1(\gamma(c)) = f_2(c)\), (33) becomes
\[
\Delta > (\gamma(c) - c) + \frac{F_1(\gamma(c))}{f_1(\gamma(c))} \left( 1 - \frac{1}{\gamma'(c)} \right).
\]
But, since \(\gamma'(0) \geq 1\), \(\gamma'\) is not the identity, \(\gamma'(c) > 1\) near 1, and so the RHS expression diverges, a contradiction. Thus there exists some \(\tilde{c} \in (\hat{c}, 1)\) such that \(\eta_M(\tilde{c}) < \gamma(\tilde{c})\). Since \(\gamma'(\tilde{c}) \geq 1\), and since \(\eta_M(1) = \gamma(1)\), it must be that for some \(c^* \in (\tilde{c}, 1)\), \(\eta_M(c^*) > 1\). ✷

**A.4.1. Proof of Theorem 6**

We begin with three lemmas.

**Lemma 19.** If \(\tilde{c}_1\) is a maximizer of \(W_{\tilde{F}_1}\) then \(\tilde{c}_1\) is also a maximizer of \(W_{\tilde{f}_1}\).

**Proof.** Assume by way of contradiction that \(\tilde{c}_1\) is not a maximizer of \(W_{\tilde{f}_1}\), then as a continuous function with compact support, \(W_{\tilde{f}_1}\) is maximized at some \(c < \tilde{c}_1\), where
\[
2 - W_{\tilde{F}_1}(c) \leq \frac{1}{W_{\tilde{f}_1}(c)}
\]
(with equality if \(c > \zeta_1\)). But since by assumption \(W_{\tilde{F}_1}\) is maximized at \(\tilde{c}_1\) we then have
\[
\frac{1}{W_{\tilde{f}_1}(c)} \geq 2 - W_{\tilde{F}_1}(\tilde{c}_1)
\]
or
\[
\frac{1}{2 - Wf_\bar{F}_1(\bar{c}_1)} \geq Wf_f_1(c).
\]

But by Lemma 11

\[
\frac{1}{2 - Wf_\bar{F}_1(\bar{c}_1)} = Wf_f_1(\bar{c}_1)
\]

and so we have \(Wf_f_1(\bar{c}_1) \geq Wf_f_1(c)\), contradicting that \(\bar{c}_1\) is not a maximizer of \(Wf_f_1\). \(\square\)

**Lemma 20.** At any point in \([\bar{c}_1, \bar{c}_1]\) where \(\frac{f'_1}{f_1} \neq 0\),

\[
\frac{(\log f_1)''}{(\log \bar{F}_1)''} = \frac{W^2 f_\bar{F}_1(1 - Wf_1)}{(1 - Wf_1)}.
\]

If \(f_1(\bar{c}_1) = 0\), then \(\frac{(\log f_1)''}{(\log \bar{F}_1)''}(\bar{c}_1)\) is well defined, and

\[
\frac{(\log f_1)''}{(\log \bar{F}_1)''}(\bar{c}_1) = Wf_\bar{F}_1(\bar{c}_1).
\]

**Proof.** Note that

\[
\frac{(\log f_1)''}{(\log \bar{F}_1)''} = \frac{\left(-\frac{f'}{f_1}\right)'}{\left(\frac{\bar{F}}{F_1}\right)'} = -\frac{f''}{f_1} + \left(\frac{f'}{f_1}\right)^2.
\]

(34)

On \([0, \bar{c}_1]\), \(\frac{\bar{F}}{F_1}\) and \(\frac{f'}{f_1}\) are well-defined. So, where \(\frac{f'}{f_1} \neq 0\), we have

\[
\frac{(\log f_1)''}{(\log \bar{F}_1)''} = \frac{-\left(\frac{f'}{f_1}\right)^2 \left(\frac{f''}{f_1^2} - 1\right)}{\left(\frac{\bar{F}}{F_1}\right)^2 \left(\frac{f'}{f_1}\right)^2 + 1} = \frac{W^2 f_\bar{F}_1(1 - Wf_1)}{(1 - Wf_1)},
\]

(35)

proving the first claim.

Assume that \(f_1(\bar{c}_1) = 0\). Then, on some interval \((\bar{c}, \bar{c}_1)\), \(f'_1 < 0\) (since, by log-concavity, \(f'_1\) crosses 0 on at most one point or interval). Hence, on this interval, (35) holds. By Lemma 12, \(Wf(\bar{c}_1) \in [0, 1]\), and, because \(f_1(\bar{c}_1) = 0\), \(Wf_\bar{F}_1(\bar{c}_1) \in \left(\frac{1}{2}, 1\right)\). But then, the RHS of (35) is continuous on \((\bar{c}, \bar{c}_1)\), and \(\frac{(\log f_1)''}{(\log \bar{F}_1)''}(\bar{c}_1)\) is well defined. From Lemma 11 with \(g = f_1\), we have, with a little rearrangement,

\[
1 - Wf_\bar{F}_1(\bar{c}_1) = \frac{1 - Wf_\bar{F}_1(\bar{c}_1)}{Wf_\bar{F}_1(\bar{c}_1)},
\]

and so

\[
\frac{W^2 f_\bar{F}_1(\bar{c}_1)(1 - Wf_\bar{F}_1(\bar{c}_1))}{1 - Wf_\bar{F}_1(\bar{c}_1)} = Wf_\bar{F}_1(\bar{c}_1). \quad \square
\]

Next, we have the obvious analog to Lemma 17.
Lemma 21. Under the conditions of Theorem 6, if \( A \leq \tau \), then FPHA\(_A\) is dominated by \( \delta \).

The proof is similar to Lemma 17, noting first that because \( \phi \) starts below \( \lambda \tau \), \( \phi \) cannot get above \( \lambda \tau \). Subject to never being above \( \lambda \tau \), \( \delta \) is the best allocation available.

Completion of the proof of Theorem 6. Given Lemma 21, we can assume \( A > \tau \). Let us show first that \( \phi' > 1 \). As before, if \( \phi' \leq 1 \) anywhere then by Lemma 18, there is an interior minimum \( r \), and by Theorem 3, we have a contradiction if \( H(r) \geq 0 \). Define

\[
K(t) = \left( \frac{1}{W_f \bar{F}_1(\phi(r))} - 2 \right) \left( \frac{f_1}{F_1}(t) - \frac{f_2}{F_2}(r) \right) + \left( \frac{f_2'}{f_2}(r) - \frac{f_1'}{f_1}(t) \right)
\]

so that

\[
H(r) = K(\phi(r)) = K(\gamma_f(\bar{r})) + \int_{\gamma_f(\bar{r})}^{\phi(r)} K'(t) \, dt.
\]

By the same argument as in the proof of Theorem 5, \( \phi(r) \geq \gamma_f(\bar{r}) \). Thus, it is enough to show that \( K(\gamma_f(\bar{r})) \geq 0 \) and that \( K'(t) \geq 0 \).

By definition, \( \frac{f_1}{F_1}(\gamma_f(\bar{r})) - \frac{f_2}{F_2}(r) = 0 \), and thus

\[
K(\gamma_f(\bar{r})) = \frac{f_2'}{f_2}(r) - \frac{f_1'}{f_1}(\gamma_f(\bar{r})).
\]

Divide \( \frac{f_2'}{f_2}(r) \) by \( \frac{f_2}{F_2}(r) \) and \( \frac{f_1'}{f_1}(\gamma_f(\bar{r})) \) by \( \frac{f_1}{F_1}(\gamma_f(\bar{r})) \) (the denominators are equal by definition), and apply (26) to conclude that \( K(\gamma_f(\bar{r})) \geq 0 \).

Now,

\[
K'(t) = \left( \frac{1}{W_f \bar{F}_1(\phi(r))} - 2 \right) \left( \frac{f_1}{F_1}(t) - \frac{f_2}{F_2}(r) \right) \left( \frac{f_1'}{f_1}(t) - \frac{f_2'}{f_2}(r) \right)
\]

where the inequality follows by A4 and the fact that \( W_f \bar{F}_1 \) is maximized at \( \bar{c}_1 \) by A3 and Lemma 19, while the last line follows by Lemmas 2 and 20.

Hence, as before, we have \( \phi' > 1 \). Consider the first crossing point \( c_3^* \) of \( \phi \) and \( \eta_M \), and let \( \tau^* = \eta_M(c_3^*) - c_2^* \). Consider first the case \( c_3^* \leq c_\tau \), illustrated in Fig. 6. Note that \( \tau^* \geq \tau \), and so, to the right of \( c_3^*, \lambda_{\tau^*} \) lies above \( \eta_M \) (since between \( c_3^* \) and \( c_\tau \), \( \eta'_M \leq 1 \), while after \( c_\tau \), \( \eta_M \leq \lambda_\tau \)). To the left of \( c_3^* \), \( \eta_M \leq 1 \). So, the SPBA implementing \( \lambda_{\tau^*} \) dominates FPHA\(_A\).
If \( c^* > c_\tau \), then everywhere before \( c_\tau \), \( \phi < \lambda_\tau \leq \eta_M \). Since \( \delta \) coincides with \( \eta_M \) to the right of \( c_\tau \), it thus follows that \( \delta \) dominates \( \phi \). We are thus done since by A5, there exists \( \hat{A} \) such that \( BS(\lambda_{\hat{A}}) \geq BS(\delta) \).53

Proof of Corollary 2. In this case, A1 is clearly satisfied and \( \eta_M \) lies strictly above \( \lambda_\tau = \gamma \) (the main diagonal) on \([0, 1]\). Thus any FPHA_A with \( A < 0 \) is dominated by FPHA_0. FPHA_0 cannot be optimal since small \( A > 0 \) distort away from the optimum on a small set of high costs, but improve the allocation on a substantially bigger set. And, since \( \eta_M \) lies above \( \gamma \), \( \eta'_M \leq 1 \), and so an SPBA implementing a line through the intersection of \( \phi \) and \( \eta_M \) is dominating as it was for Theorem 5.

Appendix B. Equivalence of auction forms

In this appendix, we formalize and prove the claim that our analysis of the first price handicap auction with two sellers translates directly to analysis of a first price handicap auction with two buyers. Let \( \hat{K} \) be the setting in which a single buyer faces two sellers with costs given by \( F_1, F_2 \), has preference \( \Delta \) for buying from 1, and is using a first price mechanism with handicap \( A \) favoring 1. Similarly, let \( \hat{K} \) be the setting in which a single seller faces two buyers with values \( G_1(s) = \tilde{F}_1(1 - s) \) on domain \([1 - \tilde{c}_1, 1 - \tilde{c}_1]\), and \( G_2(s) = \tilde{F}_2(1 - s) \) on domain \([0, 1]\), in which the seller has preference \( \Delta \) for selling to buyer 1, and is using a first price mechanism with handicap \( A \) favoring 1. In both cases, the auctions are without exclusion.

Proposition 7. Let \((\hat{\beta}_1, \hat{\beta}_2)\) and \((\beta_1, \beta_2)\) be strictly increasing and related by

\[
\hat{\beta}_1(v) = 1 - \beta_1(1 - v), \quad \text{and} \quad \hat{\beta}_2(v) = 1 - \beta_2(1 - v).
\]

Then, \((\hat{\beta}_1, \hat{\beta}_2)\) is an equilibrium of \( \hat{K} \) if and only if \((\beta_1, \beta_2)\) is an equilibrium of \( K \).

53 Note that this is the only place in the development were we resort to the weaker notion of ex-ante surplus comparisons versus dominance.
If \((\hat{\beta}_1, \hat{\beta}_2)\) and \((\beta_1, \beta_2)\) are equilibria of their respective settings, and \(\hat{\phi}\) and \(\phi\) are implicitly defined by
\[
\hat{\beta}_1(\hat{\phi}(v)) + A = \hat{\beta}_2(v),
\]
and
\[
\beta_1(\phi(v)) - A = \beta_2(v),
\]
then
\[
\hat{\phi}(v) = 1 - \phi(1 - v).
\]

Let \(\hat{\omega}_i(v)\) and \(\omega_i(c)\) be virtual valuations/costs in \(\hat{K}\) and \(K\) respectively, and let \(\hat{\mu}(v_2)\) and \(\hat{\mu}(c_2)\) define the optimal allocations in \(\hat{K}\) and \(K\) respectively. Then,
\[
\hat{\omega}_i(v) = 1 - \omega_i(1 - v),
\]
and
\[
\hat{\mu}(v_2) = 1 - \mu(1 - v_2).
\]

Thus all the relevant properties of \(K\) translate directly into properties of \(\hat{K}\).

**Proof of Proposition 7.** Let \(\hat{\phi}\) and \(\phi\) be as in (38) and (39) and let \(\hat{\psi}\) and \(\psi\) be their respective inverses. Applying (37)–(38) and simplifying yields
\[
\beta_1(1 - \hat{\phi}(v)) - A = \beta_2(1 - v).
\]
But, by (39),
\[
\beta_1(\phi(1 - v)) - A = \beta_2(1 - v),
\]
and so, since \(\beta_1\) is strictly increasing, we have (40). Similarly
\[
1 - \hat{\psi}(v) = \psi(1 - v).
\]

Arguing analogously to the proof of Theorem 1, we have for the case of \(1 - \bar{c}_1 + A \geq 0^{54}\) that if \((\hat{\beta}_1, \hat{\beta}_2)\) is an equilibrium, then the equilibrium surpluses are given by
\[
\hat{S}_1(v) = \int_{1-c_1}^{v} G_2(\hat{\psi}(s)) \, ds
\]
and
\[
\hat{S}_2(v) = \int_{A}^{v} G_1(\hat{\phi}(s)) \, ds
\]
and (following Myerson [40]) that \((\hat{\beta}_1, \hat{\beta}_2)\) form an equilibrium if and only if for all \(v \in [1 - \bar{c}_1, 1 - \xi_1]\)
\[
\hat{\beta}_1(v) = v - \frac{\hat{S}_1(v)}{G_2(\psi(v))}
\]

---

\(^{54}\text{Again, with modifications for the other case.}\)
and for all \( v \in [0, 1] \)
\[
\hat{\beta}_2(v) = v - \frac{\hat{S}_2(v)}{G_1(\phi(v))}.
\]  

(43)

Similarly, making the change of variables \( c = 1 - v \), we have that \((\beta_1, \beta_2)\) is an equilibrium if and only if for all \( v \in [1 - \bar{c}_1, 1 - \zeta_1] \)
\[
\beta_1(1 - v) = (1 - v) + \frac{S_1(1 - v)}{\bar{F}_2(\psi(1 - v))}
\]  

(44)

and for all \( v \in [A, 1] \)
\[
\beta_2(1 - v) = (1 - v) + \frac{S_2(1 - v)}{\bar{F}_1(\phi(1 - v))}
\]  

(45)

But,
\[
\hat{S}_1(v) = \int_{1-\bar{c}_1}^{v} G_2(\hat{\psi}(t)) \, dt
\]
\[
= \int_{1-\bar{c}_1}^{v} \bar{F}_2(1 - \hat{\psi}(t)) \, dt
\]
\[
= \int_{1-\bar{c}_1}^{v} \bar{F}_2(\psi(1 - t)) \, dt
\]
\[
= \int_{1-v}^{\bar{c}_1} \bar{F}_2(\psi(s)) \, ds
\]
\[
= S_1(1 - v),
\]
where the second equality holds by the definition of \( G_2 \), the third from (41), and the fourth by a change of variables.

But then, since
\[
G_2(\hat{\psi}(v)) = \bar{F}(1 - \hat{\psi}(v)) = \bar{F}(\psi(1 - v)),
\]
we have that
\[
\frac{\hat{S}_1(v)}{G_2(\hat{\psi}(v))} = \frac{S_1(1 - v)}{\bar{F}(\psi(1 - v))}
\]
and so (42) holds if and only if (44) holds. Similarly, (43) holds if and only if (45) holds.

Note next that
\[
\hat{\omega}_i(v) = v - \frac{\bar{G}_i(v)}{g_i(v)} = v - \frac{F(1 - v)}{f(1 - v)} = 1 - \omega_i(1 - v),
\]
from which it follows that \( \hat{\mu}(v_2) = 1 - \mu(1 - v_2) \) solves
\[
-\Delta = \hat{\omega}_1(\hat{\mu}(v_2)) - \hat{\omega}_2(v_2)
\]
if and only if \( \mu \) solves
\[
\Delta = \omega_1(\mu(c_2)) - \omega_2(c_2).
\]
\( \square \)
Given this, it is clear that each result in the paper has a natural analog in the setting of a seller facing two buyers. In particular, let us translate the ranking results for symmetric cost distributions. Note first that since $G_1(s) = \bar{F}_1(1 - s)$ it follows that Assumptions 2 through 4 governing our symmetric results in setting $K$ translate in setting $\hat{K}$ to

**Assumption 2.** $\rho_{\bar{G}_1}$ is decreasing.

**Assumption 3.** $\rho_{G_1}$ is minimized at $1 - \bar{c}_1$.

**Assumption 4.** $(\log g_1)^{\gamma} / (\log G_1)^{\gamma}$ is minimized at $1 - \bar{c}_1$.

Thus we have in setting $\hat{K}$ the following results equivalent to Theorem 4 and Corollary 2.

**Proposition 8.** Assume that $G_1 = G_2 = G$, that $g$ is decreasing, and $A_2^\hat{}$ is satisfied. Then, for any $A$, there is $\hat{A}$ such that $\text{SPBA}_{\hat{A}}$ strictly dominates $FPHA_A$.

**Proposition 9.** Let $G_1 = G_2 = G$ and $\Delta > 0$. Assume that $G$ satisfies $A_2^\hat{} - A_4^\hat{}$. Then the optimal $FPHA$ is dominated by an appropriately chosen $\text{SPBA}$.

In both cases $A_2^\hat{}$ ensures that $\hat{\mu}' \leq 1$, while either the assumption that $g$ is decreasing or the combination of $A_3^\hat{}$ and $A_4^\hat{}$ implies that the allocation resulting from an $FPHA$ satisfies $\hat{\phi}' > 1$. Analogs to the other results in the main body of the paper follow similarly.

**References**


[29] Vlad Mares, Jeroen Swinkels, First and second price mechanisms in procurement and other asymmetric auctions, available at: http://www.kellogg.northwestern.edu/Faculty/Directory/Swinkels_Jeroen.aspx, 2008 (an early working paper version of this paper).

